COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

UNIFORM ESTIMATES OF A TRIGONOMETRIC INTEGRAL

BY

DANIEL WATERMAN (SYRACUSE, NEW YORK)

FOR ANTONI ZYGMUND WITH RESPECT AND GRATITUDE

Let us suppose that f is a real function on the circle group T and S_n is the nth partial sum of its Fourier series. A basic tool in the study of the convergence of Fourier series is the fact that, for any $\delta > 0$,

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{\delta} (f(x+t) + f(x-t) - 2f(x)) \frac{\sin nt}{t} \, dt + o(1),$$

uniformly in x, on any interval where f is bounded [5, p. 55]. In this paper we will make uniform estimates of integrals of this form for continuous f with δ fixed and with δ varying with n.

The estimates we supply here have been used by us [4] to furnish an alternative demonstration of the result of Baernstein and Waterman [1] establishing the necessary and sufficient condition for the Fourier series of $f \circ g$ to converge uniformly for every homeomorphism g of T with itself. The first estimate of this type was used in the argument by which Goffman and Waterman [2] solved the analogous problem for everywhere convergence.

Let $\omega(f, \delta)$ denote the modulus of continuity of f. If $h = h(x,t)$ is a function on T^2, set

$$\overline{\omega}(\delta) = \overline{\omega}(h, \delta) = \sup\{|h(x,t) - h(x,t')| \mid |t - t'| < \delta, \ x \in T\}.$$

In the case

$$h(x,t) = f(x+t) + f(x-t) - 2f(x)$$

it is clear that $\overline{\omega}(h, \delta) \le 2\omega(f, \delta)$.

We introduce the notation

$$\sum(h, k, n, x, \theta) = \sum_{i=1}^{k} i^{-1} [h(x, 2i\pi/n + \theta) - h(x, (2i-1)\pi/n + \theta)].$$

THEOREM. *Let $h(x,t)$ be a function on T^2 continuous in t uniformly with respect to x, and with $h(x,0)=0$ for all x.*

(A) *There is $\theta(x,n) \in (0,\pi/n)$ such that for any sequence of positive integers $\{k_n\}$ with $\bar{\omega}(\pi/n)\log(n/k_n) = o(1)$,*

$$\int_0^\delta h(x,t)\frac{\sin nt}{t}\,dt = \frac{1}{\pi}\sum(h,k_n,n,x,\theta(x,n)) + o(1)$$

uniformly in x.

(B) *Let $\{k_n\}$ be an increasing sequence of positive integers. There is a sequence $\{\varepsilon_n\}$, $0<\varepsilon_n<\pi/n$, such that for any $h(x,t)$ there is $\theta(x,n) \in (0,\pi/n - \varepsilon_n)$ such that*

$$\int_0^{(2k_n+1)\pi/n} h(x,t)\frac{\sin nt}{t}\,dt = \frac{1}{\pi}\sum(h,k_n,n,x,\theta(x,n)) + o(1)$$

uniformly in x. The choice $\varepsilon_n = o[1/n\sqrt{\log k_n}]$ suffices for all h.

Suppose I denotes an interval $[a,b]$. By $f(I)$ we mean $f(b)-f(a)$. Applying (A) to the particular h discussed above, we obtain the

COROLLARY. *If f is a continuous function on T, then there is $\theta(x,n) \in (0,\pi/n - \varepsilon_n)$ such that for any sequence of positive integers $\{k_n\}$ with $\omega(f,\pi/n)\log(n/k_n) = o(1)$,*

$$S_n(x) - f(x) = \frac{1}{\pi^2}\left[\sum_{i=1}^{k_n} i^{-1}f(I_{ni}) - \sum_{i=1}^{k_n} i^{-1}f(J_{ni})\right] + o(1)$$

uniformly in x, where

$$I_{ni} = [x+(2i-1)\pi/n + \theta(x,n),\, x+2i\pi/n+\theta(x,n)]\,,$$
$$J_{ni} = [x-2i\pi/n - \theta(x,n),\, x-(2i-1)\pi/n-\theta(x,n)]\,.$$

The methods employed in the proofs of these results are related to those used by Salem in his work on uniform convergence [3].

Writing

$$\int_0^\delta h(x,t)\frac{\sin nt}{t}\,dt = \int_0^{\pi/n}\ldots + \int_{\pi/n}^{\delta}\ldots\quad,$$

we have

$$\left|\int_0^{\pi/n}\ldots\right| \le \varpi(h,\pi/n)\int_0^\pi\left|\frac{\sin t}{t}\right|dt = o(1)$$

as $n \to \infty$ uniformly in x. Considering the other integral,

$$\int_{\pi/n}^{\delta} \cdots = \sum_{k=1}^{N} \int_{k\pi/n}^{(k+1)\pi/n} \cdots + \int_{(N+1)\pi/n}^{\delta} \cdots ;$$

here $N+1$ is an odd integer such that $0 \leq \delta - (N+1)\pi/n < 2\pi/n$. Then

$$\left| \int_{(N+1)\pi/n}^{\delta} \cdots \right| < C\|h\|/n$$

uniformly in x, where $\|\cdot\|$ denotes the sup norm. We have

$$\sum_{k=1}^{N} \int_{k\pi/n}^{(k+1)\pi/n} h(x,t)\frac{\sin nt}{t}\,dt = \int_0^{\pi} \sum_{k=1}^{N}(-1)^k h(x,(t+k\pi)/n)\frac{\sin t}{t+k\pi}\,dt$$
$$= 2\sum_{k=1}^{N}(-1)^k \frac{h(x,(\bar{t}+k\pi)/n)}{\bar{t}+k\pi}$$

for some $\bar{t} \in (0,\pi)$. Considering two consecutive terms of this sum, k odd,

$$-\frac{h(x,(\bar{t}+k\pi/n)}{\bar{t}+k\pi} + \frac{h(x,(\bar{t}+(k+1)\pi)/n)}{\bar{t}+(k+1)\pi}$$
$$= \frac{-h(x,(\bar{t}+k\pi)/n)+h(x,(\bar{t}+(k+1)\pi)/n)}{\bar{t}+(k+1)\pi}$$
$$+h(x,(\bar{t}+k\pi)/n)\left[\frac{1}{\bar{t}+(k+1)\pi} - \frac{1}{\bar{t}+k\pi}\right]$$

and the absolute value of the last term is less than

$$\frac{\bar{\omega}((k+1)\pi/n)}{k(k+1)\pi}.$$

The sum of the absolute values of these terms from $k=1$ to q is less than $\bar{\omega}((q+1)\pi/n)/\pi$ and the sum from $q+1$ to N is less than $\|h\|/(q+1)\pi$. Choosing $q \sim \sqrt{n}$, the sum of all these terms is seen to be $o(1)$ as $n \to \infty$ uniformly in x. If in the terms of the form

$$\frac{-h(x,(\bar{t}+k\pi)/n)+h(x,(\bar{t}+(k+1)\pi)/n)}{\bar{t}+(k+1)\pi}$$

we replace the denominators by $(k+1)\pi$, observing that

$$|1/(\bar{t}+(k+1)\pi) - 1/(k+1)\pi| < 1/(k+1)^2\pi,$$

we change the sum of these terms by $O(\overline{\omega}(\pi/n))$. Thus

$$\int_{\pi/n}^{\delta} h(x,t)\frac{\sin nt}{t}\,dt = o(1) + \frac{2}{\pi}\Big[\frac{h(x,(\bar t+2\pi)/n) - h(x,(\bar t+\pi)/n)}{2} + \dots$$

$$+ \frac{h(x,(\bar t+N\pi)/n) - h(x,(\bar t+(N-1)\pi)/n)}{N}\Big]$$

$$= o(1) + \frac{1}{\pi}\Big[\sum_{i=1}^{N/2} i^{-1}(h(x,(\bar t+2i\pi)/n)$$

$$- h(x,(\bar t+(2i-1)\pi)/n))\Big].$$

Writing $\sum_{i=1}^{N/2}\dots = \sum_{i=1}^{k_n}\dots + \sum_{k_n+1}^{N/2}\dots$, we note that

$$\Big|\sum_{k_n+1}^{N/2}\dots\Big| = O(\overline{\omega}(\pi/n)\log(n/k_n)) = o(1)$$

under our hypotheses, which establishes part (A) of our theorem.

Turning now to part (B), we have

$$\int_0^{(2k_n+1)\pi/n} h(x,t)\frac{\sin nt}{t}\,dt = \int_0^{\pi/n}\dots + \sum_{i=1}^{2k_n}\int_{i\pi/n}^{(i+1)\pi/n}\dots$$

and the first integral on the right is $o(1)$ as before. As for the rest, for small $\eta_n \in (0,\pi)$, we have

$$\int_0^{\pi} \sin t \sum_{i=1}^{2k_n}(-1)^i \frac{h(x,(t+i\pi)/n)}{t+i\pi}\,dt = \int_0^{\pi-\eta_n}\dots + \int_{\pi-\eta_n}^{\pi}\dots$$

$$= (1+\cos\eta_n)\sum_{i=1}^{2k_n}(-1)^i\frac{h(x,\theta_n+i\pi/n)}{n\theta_n+i\pi} + R$$

where $\theta_n \in (0,(\pi-\eta_n)/n)$. Then

$$|R| \le \eta_n^2\frac{1}{\pi}\overline{\omega}((2k_n+1)\pi/n)\sum_{i=1}^{2k_n} i^{-1} = o(1)$$

uniformly in x as $n \to \infty$ if $\eta_n = o((\overline{\omega}((2k_n+1)\pi/n)\log k_n)^{-1/2})$. The sum is treated as in the previous case, yielding for the right hand side,

$$(1+\cos\eta_n)\frac{1}{\pi}\sum_{i=1}^{k_n}(2i)^{-1}[h(x,\theta_n+2i\pi/n) - h(x,\theta_n+(2i-1)\pi/n)] + o(1).$$

Replacing $1+\cos\eta_n$ by 2, we introduce an error bounded by

$$(1-\cos\eta_n)\frac{1}{2\pi}\overline{\omega}(\pi/n)\sum_{i=1}^{k_n} i^{-1} \le C\eta_n^2\overline{\omega}(\pi/n)\log k_n = o(1)$$

if η_n is subject to the condition above. Part (B) then follows with $\theta(x,n)\in(0,\pi/n-\varepsilon_n)$ where $\varepsilon_n=\eta_n/n$. Note that $\eta_n=o[1/\sqrt{\log k_n}]$ or, equivalently, $\varepsilon_n=o[1/n\sqrt{\log k_n}]$ suffices for our estimates and is independent of h.

REFERENCES

[1] A. Baernstein and D. Waterman, *Functions whose Fourier series converge uniformly for every change of variable*, Indiana Univ. Math. J. 22 (1972), 569–576.

[2] C. Goffman and D. Waterman, *Functions whose Fourier series converge for every change of variable*, Proc. Amer. Math. Soc. 19 (1968), 80–86.

[3] R. Salem, *Essais sur les séries trigonométriques*, Act. Sci. Ind. 862, Hermann, Paris 1940.

[4] D. Waterman, *Functions whose Fourier series converge uniformly for every change of variable II*, Indian J. Math. 25 (1983), 257–264.

[5] A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge University Press, 1959.

DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK 13244-1150, U.S.A.

Reçu par la Rédaction le 14.5.1990