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ON N. NOBLE'S THEOREMS CONCERNING POWERS OF SPACES AND MAPPINGS

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In this note we give a short proof of Noble's theorem which states that all powers of a mapping f are closed if and only if f is either perfect or constant. We derive from this theorem another Noble's theorem which states that all powers of a space X are normal if and only if X is compact.

1. For any space X and a cardinal $\mathfrak{m}$ we denote by $X^{\mathfrak{m}}$ the $\mathfrak{m}$-th power of X. We call the *$\mathfrak{m}$-th power* of the mapping f the product of $\mathfrak{m}$ copies of f and we denote it by $f^{\mathfrak{m}}$. We say that the projection $p\colon X\times Y \to Y$ is *z-closed* if it maps functionally closed sets onto closed sets. It is easy to see that if the product $X\times Y$ is normal, then the projection p is z-closed if and only if it is closed. We shall use throughout the terminology and notation of [1]. In particular, we denote by N the integers, by I the unit interval, and by $f|A$ the restriction of the mapping f to the set A.

2. We begin with stating the two theorems of Noble (see [4]).

THEOREM 1. *All powers of a space X are normal if and only if X is compact.*

THEOREM 2. *All powers of a mapping $f\colon X\to Y$, X being Hausdorff, are closed if and only if f is either perfect or constant.*

Various authors (see [2] and [3]) proposed short proofs of Theorem 1. All these proofs make use of Stone's theorem (see [1], Problem 2.7.16 (a)) which states that the product $N^{\aleph_1}$ is not normal.

3. We show now that Theorem 2 implies Theorem 1. A theorem of Tamano (see [1], Problem 3.12.20 (b)) states that the product $X\times Y$ of pseudocompact spaces X and Y is pseudocompact if and only if the projection $p\colon X\times Y\to X$ is z-closed. Suppose that all powers of the space X are normal. Consider the projection $p\colon X\times X\to X$. By the assumption, Stone's theorem and Tamano's theorem, all powers of p are closed. By Theorem 2, p is perfect. Thus X is compact.

4. Proof of Theorem 2. It was shown by Mrówka (see [1], Problem 3.12.14 (a)) that a Hausdorff space X is compact if for every compact space Y the projection $p\colon X\times Y\to Y$ is closed; clearly, it is sufficient to consider only the projections $p\colon X\times I^{\mathfrak{m}}\to I^{\mathfrak{m}}$ for $\mathfrak{m}\geqslant\aleph_0$. Suppose that all powers of a mapping $f\colon X\to Y$, where X is a Hausdorff space, are closed and that f is not constant. To show that f is perfect we consider an arbitrary fiber F of f and define an equivalence relation R on X as follows: xRy if and only if either $x\in F, y\in F$ and $x=y$ or $x\notin F, y\notin F$ and $f(x)=f(y)$.

Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ Z \arrow[r, "q_1"'] \arrow[ur, "\bar{f}"] & T \arrow[u, "h"'] \end{tikzcd}$$

where Z denotes the quotient space X/R, T denotes the quotient space Z/F, $q\colon X\to Z$ and $q_1\colon Z\to T$ are the natural mappings, and $\bar{f}\colon Z\to Y$ as well as $h\colon T\to Y$ are the mappings induced by f and $\bar{f}$, respectively. For any cardinal $\mathfrak{m}\geqslant\aleph_0$ we have $f^{\mathfrak{m}}=\bar{f}^{\mathfrak{m}}\circ q^{\mathfrak{m}}$. Thus $\bar{f}^{\mathfrak{m}}$ is closed (see [1], Proposition 2.1.3). Since $h\circ q_1=\bar{f}$, h is closed; clearly, h is one-to-one. Thus h is a homeomorphic embedding. Since $h^{\mathfrak{m}}\circ q_1^{\mathfrak{m}}=\bar{f}^{\mathfrak{m}}$, $q_1^{\mathfrak{m}}$ is closed. Now,

$$q_1^{\mathfrak{m}}\times q_1^{\mathfrak{m}}=q_3\circ q_2,\qquad\text{where } q_2=\mathrm{id}_{Z^{\mathfrak{m}}}\times q_1^{\mathfrak{m}} \text{ and } q_3=q_1^{\mathfrak{m}}\times\mathrm{id}_{T^{\mathfrak{m}}},$$

and, therefore, q_3 is closed. Since f is not constant, T contains at least two different points. T being a T_1-space, the product $T^{\mathfrak{m}}$ contains the Cantor cube $D^{\mathfrak{m}}$ as a closed subset. Consider the projection $r\colon F\times D^{\mathfrak{m}}\to D^{\mathfrak{m}}$; let $k\colon (F/F)^{\mathfrak{m}}\times T^{\mathfrak{m}}\to T^{\mathfrak{m}}$ be the projection. Since

$$r=\big(k\circ(q_3|F^{\mathfrak{m}}\times T^{\mathfrak{m}})\big)\big|F\times D^{\mathfrak{m}},$$

r is closed. Assume that $g\colon D^{\aleph_0}\to I$ maps the Cantor set onto I (see [1], Exercise 3.2.B). The mapping $g^{\mathfrak{m}}$ is closed (see [1], Theorem 3.7.7). Consider the commutative diagram

$$\begin{tikzcd} F\times D^{\mathfrak{m}} \arrow[r, "\mathrm{id}_F\times g^{\mathfrak{m}}"] \arrow[d, "r"'] & F\times I^{\mathfrak{m}} \arrow[d, "p"] \\ D^{\mathfrak{m}} \arrow[r, "g^{\mathfrak{m}}"'] & I^{\mathfrak{m}} \end{tikzcd}$$

where p is the projection. The mapping $p\circ(\mathrm{id}_F\times g^{\mathfrak{m}}) = g^{\mathfrak{m}}\circ r$ being closed, p is closed. The cardinal $\mathfrak{m}\geqslant\aleph_0$ being arbitrary, the fiber F is a compact space.

REFERENCES

[1] R. Engelking, *General topology*, Warszawa 1977.

[2] S. P. Franklin and R. C. Walker, *Normality of powers implies compactness*, Proceedings of the American Mathematical Society 36 (1972), p. 295-296.

[3] J. Keesling, *Normality and infinite product spaces*, Advances in Mathematics 9 (1972), p. 90-92.

[4] N. Noble, *Products with closed projections. II*, Transactions of the American Mathematical Society 160 (1971), p. 169-183.

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