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ON A PROBLEM OF LIPIŃSKI CONCERNING AN INTEGRAL EQUATION

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In connection with a problem raised by Gołąb [2], Lipiński [4] considered the integral

$$F(t) = \int_a^b f\big(t\varphi(u)\big)\,du, \quad 0 \leqslant t \leqslant 1, \tag{1}$$

where $\varphi(u)$ is an increasing function on $\langle a, b\rangle$ such that $\varphi(a) = 0$, $\varphi(b) = 1$, and $f(x)$ is a real-valued continuous function on $\langle 0, 1\rangle$. He showed that, even with continuous φ, the relation

$$F(t) = 0 \quad \text{for } 0 \leqslant t \leqslant 1 \tag{2}$$

does not imply the relation

$$f(x) = 0 \quad \text{for } 0 \leqslant x \leqslant 1 \tag{3}$$

and asks (P 280) whether (2) implies (3) under the additional assumption that f has a right-hand side derivative at 0. By a slight modification of his original construction we are going to show that the answer to this question is negative.

Fix a positive number k and put

$$x_n = 2^{-n}, \quad y_n = \frac{2k+1}{k+1}x_{n+1}, \quad n = 0, 1, 2, \ldots \tag{4}$$

Next define a function $\varphi(u)$ on $\langle 0, 1\rangle$ by the conditions

$$\varphi(x_n) = x_{2n}, \quad \varphi(y_n) = x_{2n+1}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

and by the requirement that φ is linear in every interval $\langle x_{n+1}, y_n\rangle$ and $\langle y_n, x_n\rangle$; moreover, $\varphi(0) = 0$. Then

$$\varphi(u) = \begin{cases} x_{2n+2} + a_n(u - x_{n+1}) & \text{if } u \in \langle x_{n+1}, y_n\rangle, \\ x_{2n+1} + b_n(u - y_n) & \text{if } u \in \langle y_n, x_n\rangle, \end{cases} \tag{6}$$

where $n = 0, 1, 2, \ldots$ and

$$a_n = (1 + k^{-1})x_{n+1}, \quad b_n = (k+1)x_n. \tag{7}$$

It is obvious from the construction that φ is a continuous and strictly increasing map of $\langle 0,1\rangle$ onto itself, and that $\varphi(0)=0$, $\varphi(1)=1$.

LEMMA 1. *Let $f(x)$ be a solution of the functional equation*

$$f(x/2) = -k^{-1}f(x), \qquad x\in\langle 0,1\rangle. \tag{8}$$

If f is continuous in $\langle 0,1\rangle$, then

$$F(t) = \int_0^1 f[t\varphi(u)]\,du = 0 \qquad \text{for } t\in\langle 0,1\rangle. \tag{9}$$

Proof. By (8) we have $f(0)=0$, i.e., (9) holds for $t=0$. For $t\in(0,1\rangle$ we first evaluate the integral

$$\int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = \int_{x_{n+1}}^{y_n} f[t\varphi(u)]\,du + \int_{y_n}^{x_n} f[t\varphi(u)]\,du.$$

By (6) and (5),

$$\int_{x_{n+1}}^{y_n} f[t\varphi(u)]\,du = \int_{tx_{2n+2}}^{tx_{2n+1}} (ta_n)^{-1} f(x)\,dx,$$

and, by (6), (5) and (8),

$$\int_{y_n}^{x_n} f[t\varphi(u)]\,du = \int_{tx_{2n+1}}^{tx_{2n}} (tb_n)^{-1} f(s)\,ds = -\int_{tx_{2n+2}}^{tx_{2n+1}} (tb_n)^{-1} 2kf(x)\,dx.$$

Hence

$$\int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = t^{-1}(a_n^{-1}-2kb_n^{-1}) \int_{tx_{2n+2}}^{tx_{2n+1}} f(x)\,dx = 0,$$

ince in view of (7) and (4) there is $(a_n^{-1}-2kb_n^{-1})=0$. Thus also

$$\int_0^1 f[t\varphi(u)]\,du = \sum_{n=0}^{\infty} \int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = 0,$$

for $t\in(0,1\rangle$, i.e. relation (9) holds true.

The following lemma is a particular case of a general theorem due to the first author of the present paper ([1]; cf. also [3], Theorem 4.9, p. 101).

LEMMA 2. *If*

$$2^r k^{-1} < 1, \tag{10}$$

then every function $g(x)$ of class C^r, defined in the interval $\langle \frac{1}{2},1\rangle$ and satisfying conditions

$$2^{-n}g^{(n)}(\tfrac{1}{2}) = -k^{-1}g^{(n)}(1) \qquad \text{for } n=0,1,\ldots,r$$

can be uniquely extended to a solution f of equation (1) *in* $\langle 0, 1\rangle$ *which is of class* C^r *in this interval.*

From Lemmas 1 and 2 we obtain the following result, which yields a negative answer to the problem P 280 of Lipiński:

THEOREM. *For every positive integer r there exist a continuous strictly increasing function φ from $\langle 0, 1\rangle$ onto $\langle 0, 1\rangle$ and infinitely many non-trivial functions f of class C^r on $\langle 0, 1\rangle$ such that relation* (9) *holds.*

In fact, for a given r we can always realize (10) by choosing k sufficiently large. Then equation (8) has in $\langle 0, 1\rangle$ a solution f of class C^r depending on an arbitrary function.

Lipiński [4] proved that for functions f analytic in $\langle 0, 1\rangle$ [1] relation (2) implies (3). This leaves the following two problems open:

P 791. Do there exist a continuous strictly increasing function φ from $\langle a, b\rangle$ onto $\langle 0, 1\rangle$ and a non-trivial function f of class C^∞ on $\langle 0, 1\rangle$ such that for function (1) relation (2) holds?

P 792. Do there exist a continuous strictly increasing function φ from $\langle a, b\rangle$ onto $\langle 0, 1\rangle$ and a non-trivial function on $\langle 0, 1\rangle$ possessing for every positive integer r the asymptotic property

$$f(x) = 0(x^r) \quad \text{for } x \to 0+0$$

such that for function (1) relation (2) holds?

We conjecture that the answer to either question is positive.

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[1] It is not enough to assume that f is analytic in $(0, 1\rangle$: e.g., function $f(x) = x^{\lg k}\sin(\pi \lg x)$, where $\lg z = \log_2 z$, is a non-trivial solution of (8), analytic in $(0, 1\rangle$.

REFERENCES

[1] B. Choczewski, *Investigation of the existence and uniqueness of differentiable solutions of a functional equation*, Annales Polonici Mathematici 15 (1964), p. 117-141.

[2] S. Gołąb, *P 40*, Colloquium Mathematicum 1 (1948), p. 240-241.

[3] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.

[4] J. S. Lipiński, *Sur une intégrale*, Colloquium Mathematicum 7 (1959), p. 67-74.

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