COLLOQUIUM MATHEMATICUM

VOL. LIX 1990 FASC. 2

CLOSED PARALLEL 1-FORMS ON INFRANILMANIFOLDS AND CALABI REDUCTIONS

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0. Introduction. In this paper we prove that any closed infranilmanifold M with

$$\mathrm{bz}_1(M) = \operatorname{rank}\operatorname{im}[Z(\pi_1(M)) \to H_1(M)] > 0$$

is affinely equivalent to $(F\times[0, 1])/((\alpha(x), 0) \sim (x, 1))$, where F is an affine submanifold of M and $\alpha\colon F\to F$ is an affine diffeomorphism of finite order (see Theorem 1.1 for a more precise statement). The affine equivalence is associated with every choice of an epimorphism $\varepsilon\colon \pi_1(M)\to Z$ with $\varepsilon(Z(\pi_1(M)))\neq 0$ (see Theorem 3.1 below).

Theorem 4.1 is an equivariant version of Theorem 1.1. It asserts the following. If a compact Lie group H acts affinely on M and $\mathrm{Fix}(H; M)\neq\emptyset$, then there is an invariant affine submanifold $F\subset M$ and an affine equivariant diffeomorphism from $F\times(0, 1)$ onto $M-F$ iff H fixes a nontrivial element of $Z(\pi_1(M))$.

When a manifold M is flat, $\mathrm{bz}_1(M)$ coincides with the first Betti number of M (see Corollary 1.1). Hence Calabi's theorem (see, e.g., [2] and [15], Theorem 3.6.3) is a particular case of Theorem 1.1.

Theorems 1.1 and 4.1 will be derived from Proposition 2.1 describing the set $A^1_{\mathrm{CP}}(M)$ of all closed and parallel 1-forms on a compact infranilmanifold M, ∇, where ∇ is the canonical connection on M. Proposition 2.1 is a simple consequence of its particular case proved by Nomizu in [11]. Note that the corresponding result is false for higher cohomology groups (see Proposition 2.3 and [11], p. 538).

1. The generalized Calabi reduction.

DEFINITION 1.1. Let M, ∇ be a manifold with connection ∇. Then by an *affine Calabi reduction* we mean an affine diffeomorphism (that is, a connection preserving diffeomorphism) $\phi\colon M\to F_\alpha$, ∇', where F is a manifold,

$$F_\alpha = \frac{F\times[0, 1]}{(\alpha(x), 0)\sim(x, 1)},$$

∇' is the connection induced by a product connection on $F \times \mathbf{R}$, and $\alpha: F \to F$ is an affine diffeomorphism of finite order r.

Remark 1.1. In a similar way one can define a *topological Calabi reduction.*

Remark 1.2. (a) An affine Calabi reduction $\phi: M \to F_\alpha$ determines a fibration $p: M \to S^1$ given by $p([x, t]) = [t] \in \mathbf{R}/\mathbf{Z} = S^1$. Here $[x, t]$ is the class of $(x, t) \in F \times [0, 1]$ in F_α, and $[t]$ is the class of $t \in \mathbf{R}$ in $\mathbf{R}/\mathbf{Z}$.

(b) As $F_{\alpha^r} = F_{\mathrm{id}} = F \times S^1$, an r-fold cover of M is affinely diffeomorphic to $F \times S^1$.

DEFINITION 1.2. If

$$\pi: \pi_1(M) \to \pi_1(M)/[\pi_1(M), \pi_1(M)] = H_1(M)$$

is the projection and $Z(\pi_1(M))$ is the center of $\pi_1(M)$, then

$$\mathrm{bz}_1(M) = \operatorname{rank} \pi\big(Z(\pi_1(M))\big).$$

One of the basic results in the flat manifolds theory is the so-called *Calabi reduction theorem.* It asserts that a closed flat manifold with the first Betti number $\mathrm{b}_1(M)$ greater than zero admits an affine Calabi reduction (see [2] and [15], Section 3.6; see also Corollary 1.1 below). This theorem allows us to study flat manifolds inductively using the induction on their dimensions.

Let G be a nilpotent, connected, simply connected Lie group and let ∇_0 be a connection on G such that left translations in G act on TG as parallel translations (cf. [1], Section 7.2, and [7], Section 2). An *infranilmanifold* is an orbit space $M = (G, \nabla_0)/\Gamma$, where Γ is a discrete group acting affinely, freely, and properly discontinuously on G, ∇_0 and such that $G \cap \Gamma$ has finite index in Γ. We will always assume that $G \cap \Gamma$ acts on G by right translations. An infranilmanifold G/Γ is a *flat manifold* if G is isomorphic to $\mathbf{R}^n$.

A closed manifold M ($\dim M > 4$) is homeomorphic to an infranilmanifold iff $\pi_1(M)$ is a finite extension of a nilpotent, torsion free, finitely generated group and $\pi_i(M) = 0$ for $i \geqslant 2$ (see [6], Theorem 5.1).

It is easy to verify that one-parameter subgroups of G are exactly geodesics and $f \in \mathrm{Aff}(G)$ (where $\mathrm{Aff}(G)$ denotes the group of all affine diffeomorphisms of G) if and only if $f = L_g \circ A$, where $g \in G$, $L_g(y) = gy$ for $y \in G$, and A is an automorphism of G (cf. [7], Section 2). Hence affine submanifolds of G, ∇ are exactly algebraically affine submanifolds of G. By an *algebraically affine submanifold of* G we mean a submanifold $L_h(H) \subset G$, where H is a Lie subgroup of G and $h \in G$.

The following result generalizes Calabi reduction to infranilmanifolds.

THEOREM 1.1. *Let M, ∇ be a closed infranilmanifold with a flat canonical connection ∇. Then there is an affine Calabi reduction of M if and only if* $\mathrm{bz}_1(M) > 0$.

Remark 1.3. (a) It is easy to see that the existence of a topological Calabi reduction implies $\mathrm{bz}_1(M) > 0$. So we should prove only the converse implication.

(b) The fact that the existence of a topological Calabi fibration implies the existence of an affine Calabi fibration seems to be a very special property of infranilmanifolds. It can fail even for nonpositive curvature manifolds. To be more specific, for every closed manifold M admitting a metric of nonpositive curvature and satisfying $Z(\pi_1(M)) \neq \{1\}$ there is (see Theorem 3.1 in [5] and [9]) a topological Calabi fibration but not always there is an affine Calabi fibration. See [9], p. 222, for a counterexample.

(c) The fact that $\phi: M \to F_\alpha$ is a Calabi reduction of an infranilmanifold M means that F is covered by an algebraically affine submanifold $\tilde{F} = L_h(H)$ (where H is a Lie subgroup of G and $h \in H$) of G, G is isomorphic to $H \times \boldsymbol{R}$, and ϕ is covered by an algebraically affine diffeomorphism $\tilde{\phi}: G \to G$. Hence Theorem 1.1 can be treated as a purely algebraic result.

(d) A topological (or smooth) version of Theorem 1.1 can be found in [5] (Theorem 3.1) and [10] (Section 4.7, Corollary 1). In [10], Section 4.7, topological Calabi fibrations are treated as a particular case of a more general notion of a Seifert fiber space with typical fiber S^1 and Corollary 1 there follows from a more general result ([10], Section 4.7, Theorem 1).

COROLLARY 1.1. *If M is a closed riemannian flat manifold, then* $\mathrm{bz}_1(M) = \mathrm{b}_1(M)$. *Hence Calabi's theorem is a particular case of Theorem* 1.1.

Proof. The proof of Corollary 1.1 is implicitly described in the proof of the Calabi reduction (see, e.g., [15], Section 3.6). For convenience of the reader we present here a short direct argument. The flatness of M implies that harmonic forms on M are parallel. Every parallel 1-form ω determines a parallel vector field X_ω that is perpendicular to $\ker\omega$ and satisfies $\omega(X_\omega(x)) = 1$ for $x \in M$. It is well known that $I_0(M)$, the identity component of the isometry group of a closed flat manifold M, is generated by parallel fields. Since $I_0(M)$ is a compact, commutative (as parallel fields commute) Lie group, $I_0(M)$ is a torus $\boldsymbol{T}^k$ and our argument shows that $k \geqslant \mathrm{b}_1(M)$. In particular, there is an S^1-action $\phi_t: M \to M$, $t \in [0, 1]$, generated by a parallel vector field X.

It is well known (see [4], Lemma 4.2) that every orbit of our S^1-action determines a central element of $\pi_1(M)$. As the vector field X is parallel, the 1-form ω, given by $\omega(v) = \langle X, v\rangle$, is parallel and, in particular, it is harmonic. For a fixed orbit $c: [0, 1] \to M$, $c(t) = \phi_t(x)$, we have

$$\int_c \omega = \int_0^1 \omega(dc/dt)\,dt = \int_0^1 \omega(X)\,dt = |X|^2 > 0.$$

This holds for any S^1-action embedded into our $\boldsymbol{T}^k$-action. Hence we have a monomorphism

$$\pi_1(\boldsymbol{T}^k) \xrightarrow{\mathrm{ev}_*} H_1(M) \to \boldsymbol{Z}^k,$$

where ev_* is induced by ev: $T^k \to M$, $\mathrm{ev}(t) = tx_0$ (here x_0 is a chosen point of M), and where the second map is induced by the projection onto the quotient of $H_1(M)$ by its torsion subgroup. As

$$\mathrm{im}\,\mathrm{ev}_* \subset \mathrm{im}\left[Z(\pi_1(M)) \to H_1(M)\right],$$

we have $\mathrm{bz}_1(M) \geqslant k$. This completes the proof of Corollary 1.1.

EXAMPLE 1.1. The last statement, that $\mathrm{b}_1(M) = \mathrm{bz}_1(M)$ for a flat manifold M, is false for infranilmanifolds. A well-known counterexample is the following. Let G be the Heisenberg group, i.e., the group of real 3×3 upper triangular matrices with diagonal entries equal to 1. It is the simplest noncommutative nilpotent Lie group. Let Γ be its subgroup of all matrices with integer entries and let $M = G/\Gamma$. Then G is diffeomorphic to $\boldsymbol{R}^3$, $\Gamma \approx \pi_1(M)$, and it is easy to see that $Z(\Gamma) = [\Gamma, \Gamma] \approx \boldsymbol{Z}$. Hence $\mathrm{bz}_1(M) = 0$. However,

$$H_1(M) \approx \Gamma/[\Gamma, \Gamma] \approx \boldsymbol{Z} \oplus \boldsymbol{Z} \quad \text{and} \quad \mathrm{b}_1(M) = 2.$$

2. Closed parallel 1-forms on infranilmanifolds. In this section we prove the following

PROPOSITION 2.1. *Let $M = G/\Gamma$ be a closed infranilmanifold and let ∇ be a canonical connection on M. Then every cohomology class $v \in H^1(M; \boldsymbol{R})$ contains a unique ∇-parallel 1-form. In particular,* $\dim A^1_{\mathrm{CP}}(M) = \mathrm{b}_1(M)$.

Throughout this section the following notation will be used. By $L(G)$ we denote the Lie algebra of a Lie group G. If H is a group (or if H is a Lie algebra), then H^{ab} denotes its abelianization, i.e., $H^{\mathrm{ab}} = H/[H, H]$, and $Z(H)$ stands for the center of H. By $B^k(G)$ we denote the set of all bi-invariant k-forms on a Lie group G, and by $\pi^*\colon B^k(G^{\mathrm{ab}}) \to B^k(G)$ the homomorphism induced by the canonical projection $\pi\colon G \to G^{\mathrm{ab}}$. The set of all parallel and closed k-forms on an infranilmanifold M, ∇ (where ∇ is a canonical connection) will be denoted by $A^k_{\mathrm{CP}}(M)$.

We will need some known facts (cf. [11], Section 4, and [12], Appendix 2) concerning closed and left-invariant 1-forms on Lie groups. They can be stated as follows.

PROPOSITION 2.2. (a) *Let G be a connected Lie group and let ω be a left-invariant 1-form on G. The following conditions are equivalent:*

(i) *the form ω is bi-invariant;*

(ii) *the form ω is closed;*

(iii) *the homomorphism $L(G) \to \boldsymbol{R} \approx L(\boldsymbol{R})$ determined by ω is a Lie algebra homomorphism.*

(b) *The homomorphism $\pi^*\colon B^1(G^{\mathrm{ab}}) \to B^1(G)$ is an isomorphism.*

The implication (i) $\Rightarrow$ (ii) is known (see, e.g., [12], Appendix 2, Section 2). The equivalence (ii) $\Leftrightarrow$ (iii) follows immediately from the formula $2d\omega(X, Y) = \omega([X, Y])$, where $X, Y \in L(G)$. If (iii) holds, then $\omega\colon L(G) \to L(\boldsymbol{R})$ has a unique factorization $\omega = \omega_0 \circ P$, where $P\colon L(G) \to L(G^{ab})$ is the canonical projection and $\omega_0\colon L(G^{ab}) \to L(\mathbf{R})$. Note that $P = d\pi$ and $\omega = \pi^* \omega_0$. Since the form ω_0 is bi-invariant, the form ω is bi-invariant as well.

COROLLARY 2.1 (see [13], Corollary 7.28, and [11]). *If $M = G/\Gamma$ is a closed nilmanifold, then*

$$\dim A^1_{\mathrm{CP}}(M) = \dim B^1(G) = \dim G^{ab} = \operatorname{rank} \Gamma^{ab} = \operatorname{rank} \pi_1(M)^{ab} = \mathrm{b}_1(M).$$

Proof. Every 1-form $\omega \in A^1_{\mathrm{CP}}(M)$ is covered by a bi-invariant 1-form and every bi-invariant 1-form can be projected onto M. Hence $\dim A^1_{\mathrm{CP}}(M) = \dim G^{ab}$. Now it suffices to check that $\dim G^{ab} = \operatorname{rank} \Gamma^{ab}$. This is well known (see [13], Theorem 2.1).

Proof of Proposition 2.1. The manifold M can be written as the orbit space $\hat{M}/A$, where $\hat{M}$ is a nilmanifold and A is a finite group acting affinely and freely on $\hat{M}$. Let $q\colon \hat{M} \to M$ be the canonical projection. By Theorem 1 of [11] (and by Proposition 2.2) the cohomology class $q^* v$ is represented by a *unique* form $\hat{\omega} \in A^1_{\mathrm{CP}}(\hat{M})$.

It suffices to show that the form $\hat{\omega}$ is A-invariant. In order to prove this note that the cohomology class $q^* v$ is A-invariant. Consider

$$\hat{\eta} = (1/|A|) \sum_{a \in A} a^* \hat{\omega}.$$

As $A \subseteq \mathrm{Aff}(M)$, we have $\hat{\eta} \in A^1_{\mathrm{CP}}(\hat{M})$. As $\hat{\eta}$ and $\hat{\omega}$ belong to the same cohomology class, $\hat{\eta} = \hat{\omega}$. The homomorphism $q^*\colon H^1(M; \mathbf{R}) \to H^1(\hat{M}; \mathbf{R})$ is a monomorphism, because the covering q is finite. If $\omega, \omega_1 \in A^1_{\mathrm{CP}}(M) \cap v$, then, by Theorem 1 of [11] again, $q^* \omega = q^* \omega_1$, so that $\omega = \omega_1$. This proves the proposition.

Remark 2.1. If an infranilmanifold M is a closed flat manifold, then harmonic forms on M are parallel. Hence, by the Hodge theorem, any cohomology class $v \in H^*(M; \mathbf{R})$ contains a unique parallel form. This cannot be extended even to nilmanifolds (cf. [11], p. 538).

PROPOSITION 2.3. *Let $M = H/\Gamma$ be the Heisenberg manifold* (see Example 1.1). *Then $A^2_{\mathrm{CP}}(M)$ is a one-dimensional vector space. All forms belonging to $A^2_{\mathrm{CP}}(M)$ are exact.*

Proof. The Lie algebra $R(H)$ of right-invariant vector fields is generated by the vector fields X, Y, Z such that $[X, Z] = [Y, Z] = 0$, $[X, Y] = Z$. The field Z is bi-invariant. For any right-invariant 1-form η and for any $U, V \in R(H)$ we have $2d\eta(U, V) = \eta([U, V])$. Hence the form $d\eta$ is bi-invariant. It determines a 2-form $\omega \in A^1_{\mathrm{CP}}(M)$. This form is exact, because a right-invariant form η can be projected onto M (the group Γ acts on H by right translations).

The fact that $\dim A^2_{\mathrm{CP}}(M) = 1$ follows from the results of [11] (see [11], p. 538, and the proof of Theorem 2 there). The proof of Proposition 2.3 is complete.

3. Proof of Theorem 1.1. Let M be a closed infranilmanifold with $\mathrm{bz}_1(M) > 0$, let $\pi\colon \pi_1(M) \to H_1(M)$ be the canonical projection, and let $\sigma \in Z(\pi_1(M)) - \{1\}$ be such that $\pi(\sigma) \neq 0$. Our proof of Theorem 1.1 is based on the existence of an affine S^1-action $\phi_t\colon M \to M$, $t \in [0, 1]$, whose orbit containing the base point belongs to σ (see [8], Section 4.3). We show that for any epimorphism $h\colon \pi_1(M) \to \pi_1(S^1)$ satisfying $h(\sigma) \neq 0$ there is an affine S^1-equivariant Calabi fibration $p\colon M \to S^1$ such that $p_* = h$. Here by an S^1-*equivariant fibration* we mean a fibration p such that $p(\phi_t(x)) - p(x)$ depends on t only.

Theorem 1.1 is a particular case of the following more technical result:

THEOREM 3.1. *Let* $M = G/\Gamma$ *be a closed infranilmanifold, let* $\sigma \in Z(\pi_1(M))$, *and let an* S^1-*action* $\phi_t\colon M \to M$, $t \in [0, 1]$, *be as above. Let* $h\colon \pi_1(M) \to \pi_1(S^1) \approx \mathbf{Z}$ *be any epimorphism such that* $h(\sigma) = r \in \mathbf{Z} - \{0\}$, *let* $\Delta = \ker h$, $N = G/\Delta$, *and let* $\bar{\phi}_t\colon N \to N$, $t \in \mathbf{R}$, *denote the* $\mathbf{R}$-*action covering our* S^1-*action on* M. *Then there are an affine submanifold* $V \subset M$ *and an affine diffeomorphism* $\alpha\colon V \to V$ *such that*

(a) *the mapping* $\psi\colon V \times \mathbf{R} \to N$ *given by* $\psi(u, t) = \bar{\phi}_t(u)$ *is an* $\mathbf{R}$-*equivariant affine diffeomorphism*;

(b) *the group of covering transformations of the covering* $q\colon N \to M$ *is infinite cyclic, and if* δ *denotes its generator, then, under the above identification of* $V \times \mathbf{R}$ *with* N,

$$\delta(u, t) = (\alpha(u), t + 1/r);$$

(c) $\alpha^r = \mathrm{id}_V$;

(d) *the fibration* $p\colon M \to S^1$ *determined by our twisted decomposition* $M = (V \times \mathbf{R})/\langle\delta\rangle = V_\alpha$ (see Remark 1.2 (a)) *satisfies* $p_* = h$; *its typical fiber is affinely diffeomorphic to* V.

Remark 3.1. A smooth variant of Theorem 3.1 (valid for all closed manifolds and for all smooth homologically injective S^1-actions) can be found in [5], Theorem 4.2.

Proof of Theorem 3.1. The homomorphism h determines a homomorphism $h_0 \in \mathrm{Hom}(H_1(M; \mathbf{Z}); \mathbf{Z})$ and h_0 determines an element

$$[h] \in \mathrm{im}\,[H^1(M; \mathbf{Z}) \to H^1(M; \mathbf{R})]$$

characterized by the equality

$$\int_\gamma [h] = h(\gamma) \quad \text{for } \gamma \in \pi_1(M). \tag{1}$$

By Proposition 2.1 there is a unique parallel 1-form ω representing $[h]$.

The form ω is nonvanishing and has integral periods. Fix $x_0 \in M$. By [3], the map $p\colon M \to \mathbf{R}/\mathbf{Z} = S^1$ given by

$$p(x) = \int_{x_0}^{x} \omega \pmod{\mathbf{Z}} \tag{2}$$

is a well-defined fibration over S^1 such that the leaves of the foliation tangent to $\ker\omega$ are connected components of the fibers of p. The form ω is ϕ_t-invariant, because $\phi_t^*\omega$ is a parallel form cohomologous to ω (cf. Proposition 2.1). It follows that the fibration p is S^1-equivariant.

Let X be the vector field generating the S^1-action. Assume that $X(x)\in\ker\omega$ for some $x\in M$. Note that $\ker\omega$ is the bundle tangent to the fibers of p. Consider c: $[0, 1]\to M$ defined by $c(t) = \phi_t(x)$. Then $c\in\sigma$ and $(dc/dt)(t) = X(c(t))$. By (1) and by the S^1-invariance of X and ω we have

$$0 \neq h(\sigma) = \int_c \omega = \int_0^1 \omega(X)\,dt = \omega(X(x)) = 0.$$

This contradiction shows that the fibers of p are transversal to the orbits of the S^1-action. Note that

$$p(\phi_t(x))-p(x) = \int_x^{\phi_t(x)} \omega \pmod{\mathbf{Z}} = \int_0^t \omega(X)\,dt \pmod{\mathbf{Z}}$$
$$= t\omega(X(x)) \pmod{\mathbf{Z}} = th(\sigma) \pmod{\mathbf{Z}} = tr \pmod{\mathbf{Z}}.$$

Hence

(3) $$p(\phi_t(x))-p(x) = tr \pmod{\mathbf{Z}},$$

(3′) $$p_*(\sigma) = h(\sigma).$$

Let V be a fiber of p. By (3) every fiber of p can be written as $\phi_t(V)$ for some $t\in[0, 1]$. The field X is parallel (because $\phi_t(x) = xg_t = g_t x$, where $t\to g_t$ is a one-parameter subgroup of the center $Z(G)$; see [8], Section 4.3) and the bundle $\ker\omega$ is parallel. Hence it is not difficult to show that for any fiber $\phi_s(V)$ and for some $\varepsilon > 0$ the map

$$\Phi\colon V\times(-\varepsilon, \varepsilon)\to\{\phi_t(V)\colon\ t\in(s-\varepsilon, s+\varepsilon)\},$$

given by $\Phi(u, t) = \phi_{t+s}(u)$, is an affine diffeomorphism. Under this identification, p can be written as $p(u, t) = r(t+s)$. It follows that the map p is affine, as claimed.

Let q: $N\to M$ be the canonical projection, let $\bar{p}$: $N\to \mathbf{R}^k$ be the fibration covering p, let $V=\bar{p}^{-1}(0)$, and $u\in V$. By (3),

(4) $$\bar{p}(\bar{\phi}_t(y)) = tr+\bar{p}(y) \quad \text{for } y\in N.$$

Hence $\bar{p}^{-1}(s) = \bar{\phi}_{s/r}(V)$ for $s\in\mathbf{R}$. It is clear that ψ: $V\times\mathbf{R}\to V$, given by $\psi(v, t) = \bar{\phi}_t(v)$, is an $\mathbf{R}$-equivariant diffeomorphism. The diffeomorphism ψ is affine (compare the argument showing that Φ is affine).

The homomorphism h_1: $\Gamma/\Delta\to\mathbf{Z}$ determined by h is an isomorphism, because h is an epimorphism and $\Delta = \ker h$. Here Δ is canonically identified with the corresponding subgroup of Γ. It follows that Γ/Δ is generated by $\delta = h_1^{-1}(1)$. Let σ_0 be the image of $\sigma\in\pi_1(M)\approx\Gamma$ in Γ/Δ.

Let $\tilde{\phi}_t: G \to G$, $t \in \boldsymbol{R}$, be the $\boldsymbol{R}$-action covering the S^1-action. Under the canonical identification of $\pi_1(M, x_0)$ with Γ, $\tilde{\phi}_1$ corresponds to σ so that $\sigma_0 = \bar{\phi}_1$. Recall that $\gamma \in \Gamma$ is identified with the homotopy class of $P \circ c$, where c: $[0, 1] \to M$ is any curve joining x_0 to $\gamma(x_0)$, and P is the canonical projection of the universal covering space of M onto M. As $p_*(\sigma) = h(\sigma) = r$, we have $\sigma_0 = \delta^r$.

The equality $\delta = h^{-1}(1)$ implies that $\bar{p}(\delta(u)) = \bar{p}(u)+1$ for $u \in V$. Since $\bar{p}(\bar{\phi}_t(u)) = rt$, we have $\delta(V) = \bar{\phi}_{1/r}(V)$. Let $\alpha = \bar{\phi}_{1/r}^{-1} \circ \delta$. Then $\alpha(V) = V$. As $\bar{\phi}_0 = \mathrm{id}_N$ and $\bar{\phi}_t$ covers ϕ_t: $M \to M$, it follows that $\bar{\phi}_t$ commutes with δ for every $t \in \boldsymbol{R}$. Hence $\alpha \circ \bar{\phi}_t = \bar{\phi}_t \circ \alpha$ and $\alpha^r = \mathrm{id}$.

Under the identification ψ: $V \times \boldsymbol{R} \to N$, the diffeomorphism α can be written as $\alpha(u, t) = (\alpha(u), t)$. As $M = N/\langle \delta \rangle$, it follows that $q|_V$ carries V diffeomorphically onto a fiber of p, which proves (b). The group $\Delta \times \langle \sigma \rangle$ is a subgroup of Γ of index r because $\Delta \times \langle \sigma \rangle = h^{-1}(r\boldsymbol{Z})$. As $h = p_*$ on $\Delta \times \langle \sigma \rangle$, it follows that $p_* = h$. This completes the proof of Theorem 3.1.

4. Equivariant affine Calabi reductions. Our aim here is to generalize the results of the previous sections to the equivariant case. Let H be a Lie group acting on a manifold V and let α: $V \to V$ be an H-equivariant diffeomorphism. Let $[x, t]$ denote the class of $(x, t) \in V \times [0, 1]$ in V_α. For every $a \in \pi_1(M)$ the symbol $\varrho(a)$ denotes the image of a in $H_1(M, \boldsymbol{Q})$.

DEFINITION 4.1. The manifold V_α with the action of H given by $h[x, t] = [h(x), t]$ is called the *equivariant mapping torus of* α.

THEOREM 4.1. *Let H be a compact Lie group acting affinely on a closed infranilmanifold $M = G/\Gamma$ with a fixed point $*$. Then the following conditions are equivalent*:

(i) *The manifold M is H-equivariantly and affinely diffeomorphic to a mapping torus V_α of an H-equivariant, periodic, affine diffeomorphism α: $V \to V$.*

(ii) *There is $\sigma \in Z(\pi_1(M))$ such that $\varrho(\sigma) \neq 0$ and $h_*(\sigma) = \sigma$ for $h \in H$.*

Remark 4.1. The case where the manifold M is flat is easier (cf. [14], Section 1).

Proof. Assume that (i) holds. Let r be the order of α. Then ϕ_t: $M \to M$, $t \in [0, 1]$, given by $\phi_t([x, s]) = [x, s+rt]$, where $[x, s]$ is treated as the class of $(x, s) \in V \times \boldsymbol{R}$ in V_α, is an S^1-action on M. Let $\sigma \in \pi_1(M)$ be the class of the orbit of our fixed point $*$. Then we know ([4], Lemma 4.2) that $\sigma \in Z(\pi_1(M))$. As an r-fold cover of M is $V_{\mathrm{id}} = V \times S^1$, we have $\varrho(\sigma) \neq 0$. By Definition 4.1 the action of H on the orbit of $*$ is trivial.

Assume (ii). Let ϕ_t: $M \to M$, $t \in [0, 1]$, be the parallel S^1-action whose orbits belong to σ (see Section 3). Since $\varrho(\sigma) \neq 0$, there is an epimorphism μ: $\pi_1(M) \to \boldsymbol{Z}$ such that $\mu(\sigma) \neq 0$. The identity component H_0 of H acts trivially on $\pi_1(M, *)$ and $K = H/H_0$ is a finite group. Set

$$\lambda(\gamma) = \sum_{k \in K} \mu(k_*(\gamma)) \quad \text{for } \gamma \in \pi_1(M).$$

The homomorphism λ is H-equivariant and $\lambda(\sigma) = |K|\,\mu(\sigma) \neq 0$.

Take $[\lambda] \in \operatorname{im}[H^1(M; \mathbf{Z}) \to H^1(M; \mathbf{R})]$ corresponding to λ (see Section 3) and a parallel form ω representing $[\lambda]$ (see Proposition 2.1). By the H-invariance of $[\lambda]$ and by the uniqueness of ω, the form ω is H-invariant. It follows (compare the proof of Theorem 3.1) that the corresponding S^1-equivariant affine fibration $p: M \to S^1$ is H-equivariant.

By [8], Section 4.3, we have $\tilde{\phi}_t(x) = g_t x = x g_t$, where $\tilde{\phi}_t: G \to G$, $t \in \mathbf{R}$, is the action of $\mathbf{R}$ covering the S^1-action and $t \to g_t$ is a one-parameter subgroup of $Z(G)$. Let $h \in H$. Take $\tilde{h} \in \operatorname{Aff}(G)$ covering h. Let $\tilde{h} = L_u \circ \Phi$, where $u \in G$, $\Phi \in \operatorname{Aut}(G)$, and let $b_t = \Phi(g_t)$. Since H acts trivially on σ, the affine actions

$$\phi_t: M \to M, \qquad \psi_t = h \circ \phi_t \circ h^{-1}: M \to M, \qquad t \in [0, 1],$$

have homotopic orbits so that $\tilde{\phi}_1 = \tilde{\psi}_1$. A direct calculation yields

$$\tilde{\psi}_t(x) = (\tilde{h} \circ \tilde{\phi}_t \circ \tilde{h}^{-1})(x) = \Phi(g_t) x = b_t x,$$

and as $\tilde{\phi}_1 = \tilde{\psi}_1$, we have $b_1 = g_1$. It follows that $b_t = g_t$ for $t \in [0, 1]$, because $b_t, g_t \in Z(G)$. Hence $\tilde{\psi}_t(x) = g_t x = \tilde{\phi}_t(x)$ for every $x \in G$. Thus the S^1-action ϕ_t commutes with h and $h(\phi_t(*)) = \psi_t(*) = \phi_t(*)$.

Let $F = p^{-1}(0)$. As $* \in F$, we have $h(F) = F$ for $h \in H$. If $x \in M$, $h \in H$, then $x \in \phi_t(F)$ for some $t \in [0, 1]$, $p(hx) = p(h\phi_t(*)) = p(\phi_t(*)) = p(x)$, and Theorem 4.1 follows from Theorem 3.1.

REFERENCES

[1] P. Buser and H. Karcher, *Gromov's Almost Flat Manifolds*, Astérisque 81, Société Mathématique de France, 1981.

[2] E. Calabi, *Closed, locally euclidean, 4-dimensional manifolds*, Bull. Amer. Math. Soc. 63 (1957), p. 135.

[3] J. Cerf, *1-formes fermées non singulières sur les variétés compactes de dimension* 3, pp. 205–219 in: *Bourbaki Seminar*, Vol. 1980/81, Lecture Notes in Math. 901, Springer, Berlin–New York 1981.

[4] P. E. Conner and F. Raymond, *Actions of compact Lie groups on a spherical manifolds*, pp. 227–264 in: *Topology of Manifolds* (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago 1970.

[5] — *Injective operations of the toral groups*, Topology 10 (1971), pp. 283–296.

[6] F. T. Farrell and W. C. Hsiang, *Topological characterisation of flat and almost flat Riemannian manifolds M^n ($n \neq 3, 4$)*, Amer. J. Math. 105 (1983), pp. 641–673.

[7] F. Kamber and P. Tondeur, *Flat Manifolds*, Lecture Notes in Math. 67, Springer, Berlin 1980.

[8] Y. Kamishima, K. B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, Quart. J. Math. Oxford Ser. (2) 34 (1983), pp. 433–452.

[9] H. B. Lawson and S. T. Yau, *Compact manifolds of nonpositive curvature*, J. Differential Geom. 7 (1972), pp. 211–228.

[10] K. B. Lee and F. Raymond, *The role of Seifert fiber spaces in transformation groups*, pp. 367–425 in: *Group Actions on Manifolds* (Boulder, Colo., 1983), Contemp. Math. 36, Amer. Math. Soc., Providence, R.I., 1985.

[11] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. (2) 59 (1954), pp. 531–538.

[12] H. V. Pittie, *Characteristic Classes of Foliations*, Pitman, London 1976.
[13] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, Berlin 1972.
[14] M. Sadowski, *Calabi's reductions and centrally trivial actions*, to appear.
[15] J. Wolf, *Spaces of Constant Curvature*, McGraw Hill, New York 1967.

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Reçu par la Rédaction le 5.10.1988;
en version modifiée le 11.4.1989