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ON THE DETERMINATION OF AN ADDITIVE ARITHMETICAL FUNCTION BY ITS LOCAL BEHAVIOUR

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We call $f(n)$ a *completely additive function* if $f(mn) = f(m)+f(n)$ for all pairs of positive integers. Let F be the set of completely additive functions, and p_i the i-th prime number.

Let $\lambda_k(N)$ be the smallest integer K with the following property: *if $f(n)\in F$ and $f(n) = 0$ for all n in $N \leqslant n \leqslant N+K$, then $f(p_i) = 0$ for $i = 1, 2, \ldots, k$.*

We prove

THEOREM 1. *For any fixed k the inequalities*

$$\lambda_k(N) < c_1\sqrt{N} \tag{1}$$

and

$$\limsup \frac{\log \lambda_k(N)}{\sqrt{(\log N)(\log\log\log N)}} \geqslant c_2 \ (>0) \tag{2}$$

hold with suitable constants c_1 and c_2.

To derive (1) we prove the following stronger

THEOREM 2. *Suppose that $f(n)\in F$ and $f(n) = c =$ constant in $N \leqslant n \leqslant N+\lambda(N)$. Then for $\lambda(N) > 4\sqrt{N}$ we have $f(n) = c = 0$ for $n \leqslant \sqrt{N}$.*

Hence immediately follows

THEOREM 3. *If $f(n)$ and $g(n)$ are in F and, for some sequence of integers $N_1 < N_2 < \ldots$ and any $j = 1, 2, \ldots,$ we have $f(n) = g(n)$ for all n in $[N_j, N_j+4\sqrt{N_j}]$, then $f(n) = g(n)$ identically.*

First we prove Theorem 2 and then (2) in Theorem 1.

We shall start by proving Lemma (A), which is a weak form of Theorem 2.

LEMMA (A). *If $f(n) \in F$ and $f(n) = c = \text{constant}$ in $N \leqslant n \leqslant 2N$, then $f(n) = c = 0$ for all $n \leqslant 2N$.*

Indeed, in the interval $[N, 2N]$ a power of 2, say $n = 2^a$, can be found, whence $af(2) = c$ follows. Furthermore, $f(2N) = f(N) = c$ implies $f(2) = 0$, and thus $c = 0$ holds. For any $m < N$ a β can be found such that $N \leqslant 2^\beta m \leqslant 2N$, whence $0 = f(2^\beta m) = \beta f(2) + f(m) = f(m)$ follows.

Let $I_k = [N/k, (N+\lambda(N))/k]$. Using the assumption of Theorem 2, we conclude that

$$f(n) = c - f(k) \quad \text{for all } n \in I_k. \tag{3}$$

Let $\lambda(N) \geqslant 4\sqrt{N}$. Then it can be easily verified that

$$\frac{N+\lambda(N)}{k+1} > 1 + \frac{N}{k} \tag{4}$$

holds for all k in

$$[\sqrt{N}] \leqslant k \leqslant 2[\sqrt{N}]. \tag{5}$$

Furthermore, it follows from (4) that the intervals I_k and I_{k+1} contain at least one common element. Consequently, by (3), we have $f(k) = f(k+1)$, i.e., $f(k) = \text{constant}$ in (5). Using (A) we have $f(n) = 0$ for $n \leqslant \sqrt{N}$. This completes the proof of Theorem 2.

Now we prove (2). Let $K \geqslant p_k$. Let

$$a_n = \prod_{\substack{p^a \| n \\ p^a \leqslant K}} p^a, \quad b_n = \prod_{\substack{p^a \| n \\ p^a > K}} p^a, \quad n = a_n b_n.$$

Suppose that all the integers n in $N \leqslant n \leqslant N+K$ have at least one prime divisor greater than K, i.e. that $b_n > 1$. It is clear that

$$(b_{n_1}, b_{n_2}) = 1 \quad \text{for all } n_1, n_2 \in [N, N+K],\ n_1 \neq n_2. \tag{6}$$

Let $x_1, \ldots, x_k$ be arbitrary complex numbers. Then there exists a function $f(n) \in F$ for which $f(n) = 0$ in $n \in [N, N+K]$ and $f(p_i) = x_i$ for $i = 1, 2, \ldots, k$. We can construct such a function as follows: let $f(p_i) = x_i$ for $i = 1, 2, \ldots, k$ and $f(p)$ be arbitrary complex values for the other primes $p \leqslant K$. For $p > K$ we define the function $f(n)$ so as to have $f(b_n) = -f(a_n)$ for all $n \in [N, N+K]$. This is possible since $b_n > 1$ and (6) holds.

Now to have (2) it is enough to prove that for infinitely many N all integers n in

$$N \leqslant n \leqslant N + K_N, \quad K_N = \exp\left(c\sqrt{(\log N)(\log\log\log N)}\right) \tag{7}$$

have at least one prime divisor greater than K. This immediately follows from a known theorem of Rankin [1] stating that the number $N(x, y)$ of integers $n \leqslant x$ all prime factors of which do not exceed y satisfies the inequality

$$N(x, y) < x \exp\left(-\frac{\log\log\log y}{\log y} \log x + O(\log\log y)\right), \quad y \to \infty.$$

Hence it follows that $N(x, K_x) < x/2K_x$, when c is small, i.e. for infinitely many N all n in (7) have at least one prime factor greater than K_N. This completes the proof of (2).

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(1) See R. A. Rankin, *The difference between consecutive prime numbers*, Journal of the London Mathematical Society 13 (1938), p. 242-247.

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