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THE METRIC SPACE $H^p(A)$, $0 < p < 1$, AND ITS CONTAINING BANACH SPACE

BY

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1. Introduction. The Hardy space H^p, $0 < p < 1$, forms a complete metric space and not a Banach space. However, Duren et al. [3] constructed a "containing Banach space" B^p which had the same continuous linear functionals as H^p and contained H^p as a dense subspace. In another paper [6] Sarason presented a study of the space $H^p(A)$ for $1 \leqslant p < \infty$, consisting of functions analytic on an annulus A, which generalized the classical Hardy space H^p of functions analytic on the unit disk. He demonstrated that $H^p(A)$ was a Banach space which exhibited many of the fundamental properties of H^p.

Each of these situations presented an interesting extension of the classical H^p-theory. In this paper* we consider both of these extensions, that is, we assume that $0 < p < 1$ and that the basic domain of analyticity is an annulus. We define a space $H^p(A)$, $0 < p < 1$, which consists of functions analytic on an annulus A and which generalizes the Hardy space H^p. We also construct a "containing Banach space" $B^p(A)$ and give a representation of the dual space in terms of functions analytic on A, continuous on the closure of A, and satisfying certain Lipschitz conditions.

In our notation we assume that $0 < p < 1$ and that r_0 is a fixed positive number less than 1. We let U denote the unit disk, E the "exterior" disk $E = \{z\colon r_0 < |z| \leqslant \infty\}$, and $A = U \cap E$. For a function f analytic on A we often write $f = f_1 + f_2$ to indicate the Laurent decomposition of f chosen so that f_1 is analytic on U and f_2 is analytic on E with $f_2(\infty) = 0$. We write $\tilde{f}(z) = f(r_0/z)$ for a given function f. By H^p or $H^p(U)$ we mean the classical Hardy space of functions analytic on U, while $H^p(E)$ denotes the space of functions f, where $\tilde{f}$ is in H^p. Furthermore, $H^p(E)$ inherits

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the metric space structure of H^p through the correspondence $f \to \tilde{f}$. The subspace of $H^p(E)$ consisting of those functions which vanish at infinity is denoted by $H_0^p(E)$. The basic theory of Hardy spaces can be found in [2] or in [4].

2. The F-space $H^p(A)$. We define the space $H^p(A)$ as the linear space of functions f analytic on the annulus A such that $|f|^p$ has a harmonic majorant on A. The metric on the space is given by

$$\|f-g\|_{H^p(A)}^p = u(\sqrt{r_0}),$$

where u is the least harmonic majorant of $|f-g|^p$ on A. With this metric, $H^p(A)$ is an F-space, that is, a linear space with a complete translation invariant metric under which scalar multiplication is continuous (see [1], II. 1.10, p. 51). F-spaces are similar to Banach spaces; in particular, the Open Mapping Theorem, the Closed Graph Theorem, and the Principle of Uniform Boundedness hold for F-spaces. By way of comparison we note that the Hardy space H^p is an F-space and that the standard metric on H^p agrees with the metric defined by evaluation of the least harmonic majorant of $|f-g|^p$ at the origin:

$$\|f-g\|_{H^p}^p = \sup_{r<1} M_p^p(f-g, r) = M_p^p(f-g, 1) = u(0).$$

We comment that this notation is not used to suggest a norm, but to recall the classical notation involving integral means.

The space $H^p(A)$ has other characterizations suggested by the classical situation:

THEOREM 2.1. *The following sets are the same:*

$J = \{f$: *f is analytic on A*, $\sup M_p^p(f, r) < \infty\}$,

$K = \{f$: *f is analytic on A, $|f|^p$ has a harmonic majorant on A*$\}$,

$L = \{f$: *f is analytic on A, $f = f_1+f_2$ with f_1 in H^p and f_2 in $H_0^p(E)$*$\}$.

Proof. The inclusion $L \subset J$ follows from the inequality $(a+b)^p \leqslant a^p+b^p$ for $a>0$, $b>0$, and $0<p<1$. For $f = f_1+f_2$ in J, fix s between r_0 and 1, so that

$$\sup_{0<r<1} M_p^p(f_1, r) \leqslant \sup_{0<r<s} M_p^p(f_1, r) + \sup_{s<r<1} M_p^p(f, r) + \sup_{s<r<1} M_p^p(f_2, r),$$

which is finite.

Together with a similar calculation for f_2, we have $J \subset L$. The inclusion $L \subset K$ can be seen by using the inequality above and the well-known result that $|g|^p$ has a harmonic majorant on U for any g in H^p.

To prove the inclusion $K \subset L$ we use a method of Rudin [5]. Thus, if u is the harmonic majorant of $|f|^p$ and s is fixed, $r_0 < s < 1$, then $|f_1|^p \leqslant u+a$ on $s<|z|<1$ for some constant a. But $u = u_1+u_2$, where u_1 is harmonic on U and u_2 is harmonic on E. Since u_2 is bounded on $|z| \geqslant s$, we conclude that $|f_1|^p \leqslant u_1+b$ on $s \leqslant |z| < 1$ for some constant b. But this is true for all U, since $|f_1|^p$ is subharmonic on U; hence f_1 is in $H^p(U)$. Analogously, f_2 is in $H_0^p(E)$.

Recall next that the linear space direct sum of two F-spaces X and Y, denoted by $X \oplus Y$, is again an F-space when given the metric defined as the sum of the coordinate metrics (see [1], p. 89). Recall also that two metric spaces are said to be *equivalent* if there is a one-to-one linear mapping T of one onto the other such that T and T^{-1} are continuous. Similar statements are well known for Banach spaces. That $H^p(A)$ can be viewed as a direct sum proves useful in establishing some basic properties.

THEOREM 2.2. *The F-spaces $H^p(A)$ and $H^p(U) \oplus H_0^p(E)$ are equivalent. Furthermore, if $f = f_1+f_2$ is in $H^p(A)$, then, for $K = (1+\sqrt{r_0})/(1-\sqrt{r_0})$,*

$$\|f\|^p_{H^p(A)} \leqslant K[\|f_1\|^p_{H^p(U)} + \|f_2\|^p_{H^p(E)}].$$

Proof. Applying the inequality $(a+b)^p \leqslant a^p+b^p$ to $|f|^p$ we find that $|f|^p \leqslant u_1+u_2$, where u_1 is the least harmonic majorant of $|f_1|^p$ on U, and u_2 is the least harmonic majorant of $|f_2|^p$ on E. But u_1 is the Poisson integral of $|f_1(e^{it})|^p$. By letting

$$g(z) = (z+\sqrt{r_0})/(1-z\sqrt{r_0}),$$

we can write

$$u_1(\sqrt{r_0}) = u_1(g(0)) \leqslant u_1(0)[(1+\sqrt{r_0})/(1-\sqrt{r_0})].$$

Similar considerations on u_2 and $\tilde{u}_2$ lead to the inequality

$$u_2(\sqrt{r_0}) \leqslant u_2(\infty)[(1+\sqrt{r_0})/(1-\sqrt{r_0})],$$

and since $\|f\|^p_{H^p(A)}$ is dominated by the sum $u_1(\sqrt{r_0})+u_2(\sqrt{r_0})$, the inequality in the theorem follows. Thus far we have shown that the map $(f_1, f_2) \to f$ is continuous. The inverse map is continuous by the Open Mapping Theorem, and the two spaces are equivalent.

As a result of this last theorem we are able to present several immediate properties of the space $H^p(A)$ related to the classical theory of Hardy spaces.

COROLLARY 2.1. *Functions in $H^p(A)$ have non-tangential limits at almost every point of the boundary of A.*

COROLLARY 2.2. *Evaluation at a point in A is a continuous linear functional on $H^p(A)$.*

Indeed, for some constant $C(p)$ and $r = |z|$,

$$|f(z)| \leqslant C(p)\|f\|_{H^p(A)} \max[(1-r)^{-1/p}, (r-r_0)^{-1/p}].$$

COROLLARY 2.3. *Evaluation of a Laurent coefficient is a continuous linear functional on $H^p(A)$.*

COROLLARY 2.4. *For f in $H^p(A)$ and $f_r(z) = f_1(rz)+f_2(z/r)$ with* $0 < r < 1$,

$$\lim_{r\to 1} \|f_r - f\|^p_{H^p(A)} = 0.$$

3. The Banach space $B^p(A)$. Duren et al. in [3] defined B^p to be the linear space of all functions f analytic on U, satisfying

$$\|f\|_{B^p} = \int_0^1 (1-r)^{1/p-2} M_1(f, r)\,dr < \infty.$$

They were able to prove the following basic properties of B^p:

THEOREM 3.1. *The space B^p is a Banach space. Furthermore,*

(a) $|f(z)| \leqslant C(p)\|f\|_{B^p}(1-r)^{-1/p}$ *for f in B^p, and* $f(z) = o[(1-r)^{-1/p}]$;

(b) *for each f in B^p,*

$$\lim_{r\to 1} \|f_r - f\|_{B^p} = 0, \quad \textit{where } f_r(z) = f(rz) \textit{ for } r < 1;$$

(c) *H^p is dense in B^p;*

(d) $\|f\|_{B^p} \leqslant C(p)\|f\|_{H^p}$ *for each f in H^p and for some constant $C(p)$.*

The Banach space $B^p(E)$ and its subspace $B^p_0(E)$ are defined as expected by using the correspondence $f \to \tilde{f}$. Alternately, the norm is given by

$$\|f\|_{B^p(E)} = \int_{r_0}^{\infty} (r-r_0)^{1/p-2}(r_0/r^3) M_1(f, r)\,dr.$$

In defining the space $B^p(A)$ we utilize this same approach of weighting functions to control the growth of functions near the boundary of the annulus A. Thus, if we let

$$\varphi(r) = (1-r)^{1/p-2} \quad \text{for } r < 1$$

and

$$\psi(r) = (r-r_0)^{1/p-2}(r_0/r^3) \quad \text{for } r_0 < r < \infty,$$

we can define $B^p(A)$ to be the space of functions analytic on the annulus A with the norm

$$\|f\|_{B^p(A)} = \int_{r_0}^{1} \varphi\psi M_1(f, r)\,dr < \infty.$$

We first point out that functions in $B^p(A)$ can be decomposed into the sum of two functions from simpler spaces:

THEOREM 3.2. *A function f is in $B^p(A)$ if and only if $f = f_1+f_2$, where f_1 is in $B^p(U)$ and f_2 is in $B^p_0(E)$.*

Proof. Let f_1 be in $B^p(U)$, let f_2 be in $B_0^p(E)$, and $r_0 < s < 1$. Then $f = f_1 + f_2$ is analytic on the annulus A, and

$$\int_{r_0}^{1} \varphi\psi M_1(f, r)\,dr \leqslant \int_{r_0}^{s} \varphi\psi M_1(f_1, r)\,dr + \int_{s}^{1} \varphi\psi M_1(f_1, r)\,dr + \\ + \int_{r_0}^{s} \varphi\psi M_1(f_2, r)\,dr + \int_{s}^{1} \varphi\psi M_1(f_2, r)\,dr.$$

Since each of these integrals is finite, the function f is in $B^p(A)$.

Conversely, if $f = f_1 + f_2$ is in $B^p(A)$, then

$$\int_{0}^{1} \varphi M_1(f_1, r)\,dr \leqslant \int_{0}^{s} \varphi M_1(f_1, r)\,dr + \int_{s}^{1} \varphi M_1(f, r)\,dr + \int_{s}^{1} \varphi M_1(f_2, r)\,dr.$$

The first and third integrals are easily seen to be finite. The second is also finite, since if we let $q = \inf[\psi(r)\colon s < r < 1]$, then

$$\int_{s}^{1} \varphi M_1(f, r)\,dr \leqslant q^{-1} \|f\|_{B^p(A)}.$$

In a similar manner it can be shown that f_2 is in $B_0^p(E)$.

COROLLARY 3.1. $H^p(A) \subset B^p(A)$.

In order to show that $B^p(A)$ forms a Banach space, we need the following result:

THEOREM 3.3. *If F is a bounded set in $B^p(A)$, then the functions in F are uniformly bounded on compact subsets of A.*

Proof. In our notation, $\bar{A}(r, s) = \{z\colon r \leqslant |z| \leqslant s\}$ for any appropriate choice of r and s. Without loss of generality, we assume that functions in F are bounded in norm by 1, and consider the closed subannulus $\bar{A}(r', r'')$ as a typical compact set in A. In connection with $\bar{A}(r', r'')$ we define two other subannuli $\bar{A}(s', s'')$ and $\bar{A}(t', t'')$, so that we have three concentric disjoint closed subannuli of A. To do this choose $r_0 < s' < s'' < r' < r'' < t' < t'' < 1$.

Now, for each f in F we may choose a value $\hat{t}$ between t' and t'', and a value $\hat{s}$ between s' and s'', so that $M_1(f, \hat{t}) \leqslant M_1(f, r)$ for all r in the interval $[t', t'']$ and $M_1(f, \hat{s}) \leqslant M_1(f, r)$ for all r in the interval $[s', s'']$. Since the norm of f is bounded by 1 in our hypothesis, we note that

$$1 \geqslant M_1(f, \hat{t}) \int_{t'}^{t''} \varphi\psi\,dr \quad \text{and} \quad 1 \geqslant M_1(f, \hat{s}) \int_{s'}^{s''} \varphi\psi\,dr.$$

Now let $z = re^{it}$ be a point in $\bar{A}(r', r'')$ and use the Cauchy representation of $f(z)$ on the circles of radii $\hat{t}$ and $\hat{s}$. We have

$$|f(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\hat{t}e^{iw})\hat{t}e^{iw}}{\hat{t}e^{iw} - re^{it}}\, dw - \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\hat{s}e^{iw})\hat{s}e^{iw}}{\hat{s}e^{iw} - re^{it}}\, dw \right|$$
$$\leqslant M_1(f, \hat{t})(\hat{t} - r)^{-1} + M_1(f, \hat{s})(r - \hat{s})^{-1}.$$

Together with what we know of t and s, we can now write

$$|f(z)| \leqslant \left[(t' - r'') \int_{t'}^{t''} \varphi\psi\, dr\right]^{-1} + \left[(r' - s'') \int_{s'}^{s''} \varphi\psi\, dr\right]^{-1},$$

which is a constant dependent on the compact set $\bar{A}(r', r'')$ and not on the choice of the function f in F. Hence, the family F is uniformly bounded on $\bar{A}(r', r'')$.

THEOREM 3.4. *The space $B^p(A)$ is a Banach space, and convergence in the norm implies uniform convergence on compact sets of A.*

Proof. Let (f_n) be a Cauchy sequence in $B^p(A)$. Since $B^p(A)$ is contained in the L^1-space of functions on A under the measure $(1/2\pi)\varphi(r)\psi(r)dr\,dw$, which is known to be complete, we have $f_n \to f$ in L^1 mean for some f in L^1. This implies that some subsequence converges pointwise a.e. to f. On the other hand, this subsequence is bounded in the $B^p(A)$-norm; therefore, by the last theorem, it is a normal family. Thus, there is a subsequence (f_m), which converges uniformly on compact sets to some function g, analytic on the annulus A. Therefore, f and g agree a.e. Since g is analytic and in the L^1-space, we know that g is in $B^p(A)$, and that $f_n \to g$ in the $B^p(A)$-norm. We have also shown in this argument that norm convergence of (f_n) to g implies uniform convergence on compact subsets of A.

THEOREM 3.5. *The Banach spaces $B^p(A)$ and $B^p(U) \oplus B_0^p(E)$ are equivalent.*

Proof. It suffices to show that $(f_1, f_2) \to f$ is a continuous map from $B^p(U) \oplus B_0^p(E)$ to $B^p(A)$. Since the map is one-to-one, linear, and onto, the Open Mapping Theorem implies the continuity of the inverse. Furthermore, the question of continuity can be reduced to the inequalities

$$(1) \qquad \|f_1\|_{B^p(A)} \leqslant k\,\|f_1\|_{B^p(U)} \quad \text{and} \quad \|f_2\|_{B^p(A)} \leqslant k'\,\|f_2\|_{B^p(E)}.$$

These inequalities are clearly true for $p \geqslant 1/2$, since φ and ψ are bounded on the interval $(r_0, 1)$. Thus, we need to consider the case $p < 1/2$, where φ is increasing and ψ is decreasing on their respective domains.

We now turn our attention to f_1 and assume that $\|f_1\|_{B^p(A)} = 1$. The general case for arbitrary f_1 follows by applying this special case to the function divided by its norm. Since $\varphi(r)M_1(f_1, r)$ is increasing with r, we have

$$1 \geqslant \int_R^1 \varphi\psi M_1(f_1, r)\,dr \geqslant \varphi(R) M_1(f_1, R) \int_R^1 \psi\,dr$$

and

$$\varphi(R) M_1(f_1, R) \leqslant \Big[\int_R^1 \psi\,dr\Big]^{-1}$$

for any choice of R between r_0 and 1. Furthermore,

$$\int_{r_0}^R \varphi\psi M_1(f_1, r)\,dr \leqslant \Big[\int_R^1 \psi\,dr\Big]^{-1} \int_{r_0}^R \psi\,dr.$$

Now fix R so that

$$\int_{r_0}^R \psi\,dr = \frac{1}{4}\int_{r_0}^1 \psi\,dr \quad \text{and} \quad \int_R^1 \psi\,dr = \frac{3}{4}\int_{r_0}^1 \psi\,dr.$$

Thus

$$\int_{r_0}^R \varphi\psi M_1(f_1, r)\,dr \leqslant \frac{1}{3}.$$

We now can compute

$$\|f_1\|_{B^p(U)} \geqslant \int_R^1 \varphi M_1(f_1, r)\,dr \geqslant [\psi(R)]^{-1} \int_R^1 \varphi\psi M_1(f_1, r)\,dr$$
$$\geqslant [\psi(R)]^{-1}\Big(1 - \frac{1}{3}\Big) \equiv \frac{1}{k} > 0.$$

Hence $\|f_1\|_{B^p(A)} \leqslant k\|f_1\|_{B^p(U)}$.

The inequality for f_2 in (1) follows from the above argument if we consider the function $\tilde{f}_2$. Alternately, the proof could be modified by interchanging the roles of φ and ψ, using the fact that $M_1(f_2, r)$ decreases with r.

We are now in a position to gain information from the results of Duren et al. [3] on the space B^p as stated in Theorem 3.1.

COROLLARY 3.2. *Let f be in $B^p(A)$. Then*

(a) *for some $C(p)$,*

$$|f(z)| \leqslant C(p)\,\|f\|_{B^p(A)} \max[(1-r)^{-1/p}, (r-r_0)^{-1/p}],$$

and $f(z) = o\big[\max[(1-r)^{-1/p}, (r-r_0)^{-1/p}]\big]$;

(b) *for* $r < 1$,

$$\lim_{r\to 1} \|f_r - f\| \to 0, \quad \textit{where } f_r(z) = f_1(rz) + f_2(z/r);$$

(c) *$H^p(A)$ is dense in $B^p(A)$;*

(d) $\|f\|_{B^p(A)} \leqslant C(p)\|f\|_{H^p(A)}$ *for some constant $C(p)$.*

4. The dual space of $H^p(A)$ and $B^p(A)$. In this section we derive a representation of the dual space $H^p(A)^*$ in terms of functions analytic on A, continuous on $\bar{A}$, and satisfying a certain Lipschitz condition. Analogously to the study in [3] this space is the dual space of $B^p(A)$ as well as of $H^p(A)$. We first introduce the spaces of analytic functions that are useful in this representation.

If $F(t)$ is a complex-valued function defined on the real line, then the modulus of continuity of F is the function

$$\omega(h; F) = \sup\{|F(t)-F(s)|: \ |t-s| \leqslant h\}.$$

Furthermore, the function F is said to belong to the Lipschitz class Λ_a $(0 < a \leqslant 1)$ if

$$\omega(h; F) = O[h^a] \quad \text{as } h \to 0.$$

A continuous function F is said to belong to the class Λ_* if

$$|F(t+h)-2F(t)+F(t-h)| = O[h] \quad \text{for all } t \text{ as } h \to 0.$$

For any $a < 1$, we have $\Lambda_1 \subset \Lambda_* \subset \Lambda_a$. See [2], p. 71, for a more thorough discussion.

For functions f analytic on the annulus A, we say that f is in Λ_a (or in Λ_*) if f is continuous on the closed annulus $\bar{A}$ and both boundary functions $f(r_0e^{it})$ and $f(e^{it})$ are in the class Λ_a (or in Λ_*).

We define $\Lambda_a^n(A)$, for non-negative integer n and $0 < a \leqslant 1$, to be the space of functions g analytic on the annulus A with $g, g', g'', \ldots, g^{(n)}$ all analytic on the annulus A, continuous on $\bar{A}$, and with $g^{(n)}$ in the class Λ_a. We define $\Lambda_*^n(A)$ in a similar way using the class Λ_*. These spaces generalize those utilized in [3] for the study of functionals on the Hardy space H^p. In fact, we may refer to the spaces in [3] as to $\Lambda_a^n(U)$ and $\Lambda_*^n(U)$, and note that $\Lambda_a^n(E)$ and $\Lambda_*^n(E)$ may also be suitably defined. Of course, it is possible to use the Laurent decomposition to write a function g in $\Lambda_a^n(A)$ as $g = g_1+g_2$, where g_1 is in $\Lambda_a^n(U)$ and g_2 is in $\Lambda_a^n(E)$.

THEOREM 4.1. *With any Φ in $H^p(A)^*$ there is associated a unique function g, analytic on A and continuous on $\bar{A}$, such that*

$$\Phi(f) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f_1(re^{is})g(e^{-is})\,ds + \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi} f_2\left(\frac{r_0e^{is}}{r}\right)g(r_0e^{-is})\,ds \tag{2}$$

for any $f = f_1+f_2$ in $H^p(A)$. If $(n+1)^{-1} < p < n^{-1}$ for some positive integer n, then g is in $\Lambda_a^{n-1}(A)$, where $a = 1/p-n$. If $p = (n+1)^{-1}$ for some positive integer n, then g is in $\Lambda_*^{n-1}(A)$.

Conversely, if $(n+1)^{-1} < p < n^{-1}$ *and* g *is in* $\Lambda_\alpha^{n-1}(A)$, *where* $\alpha = 1/p - n$, *then the limit in* (2) *exists for all* f *in* $H^p(A)$ *and defines a continuous linear functional on* $H^p(A)$. *If* $p = (n+1)^{-1}$ *and* g *is in* $\Lambda_*^{n-1}(A)$, *then the limit in* (2) *exists for all* f *in* $H^p(A)$ *and defines a continuous functional on* $H^p(A)$.

Proof. Let Φ be in $H^p(A)^*$ and let

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

be in the space $H^p(A)$. Put

$$\begin{aligned} b_n &= \Phi(z^n) && \text{for } n = 0, 1, 2, \ldots, \\ b_{-n} &= r_0^{2n}\Phi(z^{-n}) && \text{for } n = 1, 2, 3, \ldots \end{aligned}$$

This choice enables us to construct analytic functions

$$g_1(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad g_2(z) = \sum_{n=1}^{\infty} b_{-n} z^{-n}$$

on U and on E, respectively. Next consider the function $f_r(z) = f_1(rz) + f_2(z/r)$ which is the uniform limit of its partial sums on the closure of A. By Corollary 3.2 we know that $f_r \to f$ in the $H^p(A)$-metric. Therefore, by the continuity of Φ,

$$\Phi(f) = \lim_{r\to 1^-} \Phi(f_r) = \lim_{r\to 1^-} \Big[\sum_{n=0}^{\infty} a_n b_n r^n + \sum_{n=1}^{\infty} a_{-n} b_{-n} r_0^{-2n} r^n\Big]. \tag{3}$$

If we consider f_1 and f_2 separately, then we can write

$$\Phi(f_1) = \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n b_n r^n \quad \text{and} \quad \Phi(f_2) = \lim_{r\to 1^-} \sum_{n=1}^{\infty} (a_{-n} r_0^{-n})(b_{-n} r_0^{-n}) r^n.$$

Note that $(a_{-n} r_0^{-n})$ and $(b_{-n} r_0^{-n})$ are the Taylor coefficients of $\tilde{f}_2$ and $\tilde{g}_2$, which are analytic in U, $\tilde{f}_2$ in $H^p(U)$. Therefore, from these last representations and from Theorem 1 of [3], which characterizes the continuous linear functionals on the space H^p as the limits of such sums, we know that g_1 and $\tilde{g}_2$ are in the class $\Lambda_\alpha^{n-1}(U)$ if $(n+1)^{-1} < p < n^{-1}$ and $\alpha = 1/p - n$. Thus, the function $g = g_1 + g_2$ is in the class $\Lambda_\alpha^{n-1}(A)$. A similar argument follows if $p = (n+1)^{-1}$, showing that g is in $\Lambda_*^{n-1}(A)$.

In addition, we see that this argument establishes the existence of boundary values for g_1, g_2, and g. Therefore, it is possible to rewrite the expansion in (3) as the integral expression in (2), noting that Cauchy's Theorem assures us that the integrals involving $f_1(re^{is})g_2(e^{-is})$ and $f_2(r_0e^{-s}/r)g_1(r_0e^{-is})$ are zero.

Conversely, suppose that g is in $\Lambda_a^{n-1}(A)$ with $(n+1)^{-1} < p < n^{-1}$ and $a = 1/p - n$. By Theorem 1 of [3], we know that

$$
\begin{aligned}
\Psi_1(f_1) &= \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f_1(re^{is}) g_1(e^{-is})\, ds, \\
\tilde{\Psi}_2(\tilde{f}_2) &= \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_2(re^{is}) \tilde{g}_2(e^{-is})\, ds
\end{aligned} \tag{4}
$$

both exist and define continuous linear functionals on H^p. If we put $\Psi(f) = \Psi_1(f_1) + \tilde{\Psi}_2(\tilde{f}_2)$, then it is easy to check that Ψ determines a continuous linear functional on the space $H^p(A)$. The argument can be repeated with $*$ in place of the subscript a.

As an example of this representation, let t be a point in A, and set

$$
\begin{aligned}
h_1(z) &= (1-tz)^{-1} = \sum_{n=0}^{\infty} t^n z^n, \\
h_2(z) &= r_0 (tz - r_0)^{-1} = \sum_{n=1}^{\infty} t^{-n} r_0^{2n} z^{-n}.
\end{aligned}
$$

Then the function $h = h_1 + h_2$ is analytic on the closed annulus $\bar{A}$, and h is in all of the Lipschitz classes. Furthermore, if Φ is the continuous linear functional induced by h, and f is any function in $H^p(A)$, then it can be checked from (3) that $\Phi(f) = f(t)$. In other words, h is the function that induces the functional which is evaluation at the point t.

In another use of this representation and of this same function h, let Ψ be a functional induced by some function g. Then by (3) it is easily verified that $\Psi(h) = g(t)$. In this way, h provides a means of determining the function g that induces a given linear functional Ψ. For example, if Ψ is evaluation of the n-th Laurent coefficient, then

$$
g(t) = \Psi(h) = \begin{cases} t^n & \text{if } n \text{ is non-negative}, \\ r_0^{-2n} t^n & \text{if } n \text{ is negative}. \end{cases}
$$

THEOREM 4.2. *The spaces $B^p(A)$ and $H^p(A)$ have the same continuous linear functionals; furthermore, Theorem 4.1 remains valid if $H^p(A)$ is replaced by $B^p(A)$.*

Proof. Let Φ be in $B^p(A)^*$ with

$$
g(z) = \sum_{-\infty}^{\infty} b_n z^n
$$

defined as in the proof of Theorem 4.1. From Corollary 3.2 (d) it follows that Φ is also a continuous linear functional on $H^p(A)$. Thus, g is in the proper Lipschitz class. Again set $f_r(z) = f_1(rz) + f_2(z/r)$ for any f in $B^p(A)$. Since

$f_r \to f$ in the $B^p(A)$-norm by Corollary 3.2 (b), the representations in (3) and in (2) hold just as in the proof of Theorem 4.1.

Conversely, if g is in the proper Lipschitz class, we know from Theorem 7 of [3] that the two equations in (4) both exist and define a linear functional on B^p. Since $B^p(A)$ is equivalent to $B^p(U) \oplus B_0^p(E)$, $\Psi(f) = \Psi_1(f_1) + \tilde{\Psi}_2(\tilde{f}_2)$ defines a continuous linear functional on $B^p(A)$.

REFERENCES

[1] N. Dunford and J. Schwartz, *Linear operators*, Part I, New York 1958.

[2] P. Duren, *Theory of H^p spaces*, New York 1970.

[3] — B. Romberg and A. Shields, *Linear functionals on H^p with* $0 < p < 1$, Journal für die reine und angewandte Mathematik 238 (1969), p. 255-262.

[4] K. Hoffman, *Banach spaces of analytic functions*, Englewood Cliffs, N. J., 1962.

[5] W. Rudin, *Analytic functions of class H_p*, Transactions of the American Mathematical Society 78 (1955), p. 46-66.

[6] D. Sarason, *The H^p spaces of an annulus*, Memoirs of the American Mathematical Society 56 (1965).

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