COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

HOMOGENEITY OF βG IF G IS A TOPOLOGICAL GROUP

BY

ERIC K. VAN DOUWEN (ATHENS, OHIO)

1. Introduction. The main purpose of this paper is to improve a result of Comfort and Ross [4]. Along the way we get a useful criterion for non-homogeneity. In the sequel we assume that all spaces are completely regular.

THEOREM 1. *The following conditions on a topological group G are equivalent*:

(a) *G is pseudocompact*;

(b) *βG can be given the structure of a topological group in such a way that G is canonically embedded as a subgroup*;

(c) *βG can be given the structure of a topological group*;

(d) *βG is homogeneous*;

(e) *βG is dyadic* (¹);

(f) *βG is the continuous image of a supercompact* (²) *Hausdorff space.*

We first give that part of the proof which follows immediately from results in the literature. Glicksberg [12] noted without proof that (b) implies (a). Comfort and Ross [4] proved that (a) and (b) are equivalent — an elementary proof was recently given by de Vries [16]. Implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial.

It is a deep result of Kuz'minov [15] that every compact topological group is dyadic. This proves implication (c) $\Rightarrow$ (e), while (e) $\Rightarrow$ (f) is trivial. Engelking and Pełczyński [8] proved that if βX is dyadic, then X is pseudocompact. This was recently improved by the author and van Mill [7] who showed that if βX is the continuous image of a supercompact Hausdorff space, then X is pseudocompact (see [1] for further improvements). This proves (f) $\Rightarrow$ (a) (³).

(¹) A space is *dyadic* if it is the continuous image of $\{-1, 1\}^{\varkappa}$ for some $\varkappa$.

(²) A space is *supercompact* if it has a subbase $\mathscr{S}$ such that every cover by elements of $\mathscr{S}$ has a subcover by at most 2 elements. $\{-1, 1\}^{\varkappa}$ is supercompact for all $\varkappa$, since any product of supercompact spaces is supercompact.

(³) Implications (c) $\Rightarrow$ (e) $\Rightarrow$ (a) are proved in [16], 4.6.4 (which considers (b) rather than (c)). The author is indebted to Wis Comfort for suggesting to include (e).

It remains to prove that (d) implies (a). This is true even if G is not a topological group. We will prove a criterion for non-homogeneity which implies the following

THEOREM 2. *The space X is not homogeneous if one of the following conditions is satisfied:*

(a) $X = \beta Y$ *for some non-pseudocompact* Y;

(b) $X = \beta Y - Y$ *for some non-pseudocompact* Y;

(c) *every countable relatively discrete subset of X is C^*-embedded and X has an infinite compact subset.*

Condition (a) of this theorem is new [4], (b) is due to Frolík [9], and (c) follows from [10] and [14]. Our method of proof deals basically with condition (c). Although this method has appeared in [2], it seems to be worthwhile to give a fairly detailed proof of Theorem 2; our proof is different in spirit from that of [2].

The fact that implication (d) $\Rightarrow$ (a) from Theorem 1 is true even if G is not a topological group suggests the question of whether βX is homogeneous if X is a pseudocompact homogeneous space. In Section 5 we answer this question in the negative by constructing a counterexample.

We use ω for the space of non-negative integers and also for the first infinite cardinal. Let f be a function, and A a set. Then $f^{\rightarrow}A$ and $f^{\leftarrow}A$ denote the image and the preimage, respectively, of A under f. If f is a bijection, then f^{-1} is the inverse function. If $f\colon X \to Y$ is a function, then $\beta f\colon \beta X \to \beta Y$ is its Stone extension.

2. The criterion. The proof of Theorem 2 (b) and (c) suggests the following concept:

Definition. A subset S of a space X is called *shy* if for all $A \subseteq S$ and for every countable, discrete, C^*-embedded $B \subseteq X$ the condition $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ implies $\bar{A} \cap \bar{B} = \emptyset$.

We then have the following criterion for non-homogeneity:

LEMMA 1. *A space is non-homogeneous if it has a countably infinite, discrete, C^*-embedded shy subset whose closure is compact.*

It is clear that Theorem 2 (c) follows from Lemma 1. To see that Theorem 2 (a) and (b) also follow from Lemma 1, it clearly suffices to prove the following

LEMMA 2. *If X is not pseudocompact, then βX has a countably infinite, discrete, C^*-embedded shy subset S such that $\bar{S} \subseteq \beta X - X$.*

We prove Lemmas 1 and 2 in Sections 3 and 4, respectively.

An immediate corollary to Lemma 1 is that every infinite compact F-space is non-homogeneous. Another corollary is that the set $\varrho(\boldsymbol{R})$

[4] It seems that this has not been considered before, probably because it is trivial for spaces X like ω, $\boldsymbol{Q}$ and $\boldsymbol{R}$.

of *remote points* of $\beta \boldsymbol{R}$ is non-homogeneous. Indeed, it is shown in [6] that $\varrho(\boldsymbol{R})$ is an infinite space in which every countable subset has compact closure; moreover, every countable subset of $\varrho(\boldsymbol{R})$ is C^*-embedded since $\varrho(\boldsymbol{R}) \subseteq \beta\boldsymbol{R} - \boldsymbol{R}$. This justifies a claim in [6].

3. Shyness and non-homogeneity. In this section we prove Lemma 1.

Let X be a space, let $x \in X$, and let $\langle e_n \rangle_{n\in\omega}$ be a sequence of points of X with $e_n \neq x$ for $n \in \omega$. Then one can define the *way* $\langle e_n \rangle_{n\in\omega}$ *clusters at* x (cf. [5]) to be the set

$$W = \{a \subseteq \omega \colon x \in \{e_n \colon n \in a\}^-\}.$$

If $\{e_n \colon n \in \omega\}$ is a discrete C^*-embedded subset of X and $e_m \neq e_n$ whenever $m \neq n$, and $x \in \{e_n \colon n \in \omega\}^-$, then, clearly, W is a free ultrafilter on ω, i.e. it is a point of $\beta\omega - \omega$. So, if we put

$$w(x, X) = \{p \in \beta\omega - \omega \colon \text{there is an embedding } e \colon \beta\omega \to X \text{ with } e(p) = x \text{ and } e^{\to}\omega \text{ is shy}\},$$

then $w(x, X)$ can be thought of as the set of all possible ways shy discrete C^*-embedded sequences with compact closures cluster at x. This is of course closely related to the types of Frolík [9], p. 707.

It is clear that $w(x, X) = w(y, X)$ whenever there is a homeomorphism of X onto itself which maps x onto y. So

(1) X is not homogeneous if there are $x, y \in X$ with $w(x, X) \neq w(y, X)$.

If the condition of (1) is satisfied, then there is an embedding $e \colon \beta\omega \to X$ with $e^{\to}\omega$ shy, and so

$$\bigcup\{w(x, X) \colon x \in X\} = \beta\omega - \omega.$$

Consequently,

LEMMA 3. *If there is an* $x \in X$ *such that* $\emptyset \neq w(x, X) \neq \beta\omega - \omega$, *then* X *is not homogeneous.*

Note that Lemma 3 and condition (1) are equally strong; however, Lemma 3 is more manageable.

Lemma 1 is an immediate consequence of Lemma 3 and of the following

LEMMA 4. $w(x, X) \neq \beta\omega - \omega$ *for all* X *and for all* $x \in X$.

For the proof of this lemma we need some facts about $\beta\omega$. There are two useful preorders on $\beta\omega$:

the *Rudin-Frolík preorder* $\sqsubseteq$ defined by $p \sqsubseteq q$ if $p \in w(q, \beta\omega)$ or, equivalently, if there is an embedding $e \colon \beta\omega \to \beta\omega$ with $e(p) = q$;

the *Rudin-Keisler preorder* $\preceq$ defined by $p \preceq q$ if there is a map $f \colon \omega \to \omega$ with $\beta f(p) = q$.

The following lemma summarizes relevant information about these preorders.

LEMMA 5. *Let* $p, q, r \in \beta\omega$.

(a) *If* $p \sqsubseteq q$, *then* $p \preceq q$ ([3], 16.12 (a)).

(b) $p \preceq q$ *and* $q \preceq p$ *iff there is a homeomorphism from* $\beta\omega$ *onto itself which maps* p *onto* q ([3], 9.3).

(c) *If* $p \sqsubseteq r$ *and* $q \sqsubseteq r$, *then* $p \sqsubseteq q$ *or* $q \sqsubseteq p$ ([3], 16.16).

(d) *There are* $p, q \in \beta\omega - \omega$ *such that* p non $\preceq q$ *and* q non $\preceq p$ [14].

(e) $|\{p \in \beta\omega - \omega: p \preceq q\}| \leqslant 2^\omega$ *for all* $q \in \beta\omega - \omega$ ([3], p. 206, (b)).

Part (a) is easy to prove, and part (d) is a highly non-trivial result of Kunen [14]. The following lemma, which generalizes part (c), implies Lemma 4 because of (a) and (d) of Lemma 5, and this will complete the proof of Lemma 1.

LEMMA 6. *Let* $x \in X$. *Then* $p \sqsubseteq q$ *or* $q \sqsubseteq p$ *for all* $p, q \in w(x, X)$.

Before we proceed to the proof we observe the following

FACT. *If* $p \in \beta\omega - \omega$ *and* $p \in \bar{A}$ *for some* $A \subseteq \omega$, *then there is a bijection* $b: \omega \to A$ *such that* $\beta b(p) = p$.

Indeed, let A_0 and A_1 be infinite disjoint subsets of A with $A_0 \cup A_1 = A$. Then there is an $I \in \{A_0, A_1\}$ with $p \in \bar{I}$. Let $b: \omega \to A$ be any bijection such that $b(i) = i$ for $i \in I$.

We now are ready for the proof of Lemma 6.

Let $f, g: \beta\omega \to X$ be embeddings with

(2) $\quad f(p) = g(q) = x,$

for which $f^{\to}\omega$ and $g^{\to}\omega$ are shy. Note that

(3) $\quad$ every subset of $f^{\to}\omega$ or of $g^{\to}\omega$ is discrete and C^*-embedded.

Put

$$F = f^{\to}\omega - g^{\to}\beta\omega \quad \text{and} \quad G = g^{\to}\omega - f^{\to}\beta\omega.$$

Then $\bar{F} \cap G = F \cap \bar{G} = \emptyset$. Hence, by (3), $\bar{F} \cap \bar{G} = \emptyset$ since $g^{\to}\omega$ is a shy subset of X. We may assume therefore that $x \notin \bar{F}$ (and will conclude $p \sqsubseteq q$). Let

$$K = f^{\to}\omega \cap g^{\to}\beta\omega.$$

Since $x \in f^{\to}\beta\omega = (f^{\to}\omega)^-$, our assumption implies that $x \in \bar{K}$. Since $f: \beta\omega \to f^{\to}\beta\omega$ is a homeomorphism, it follows from the Fact that there is a bijection $b: \omega \to K$ such that $\beta b(p) = f(p) = x$. Let us denote βb by h. Then h is a homeomorphism, and $h^{\to}\beta\omega \subseteq g^{\to}\beta\omega$ since $h^{\to}\omega = K \subseteq g^{\to}\beta\omega$. As $h(p) = x = g(q)$, it follows that $(g^{-1}|h^{\to}\beta\omega) \circ h$ is an embedding of $\beta\omega$ into $\beta\omega$ which maps p onto q, i.e. $p \sqsubseteq q$.

This completes the proof of Lemma 6, and hence of Lemma 1.

Remark. Instead of our $w(x, X)$ one can define $\tau(x, X) \subseteq \beta\omega - \omega$ by

$$\tau(x, X) = \{p \in \beta\omega - \omega: \text{there is an embedding } e: \beta\omega \to X \text{ with } e(p) = x\}.$$

Frolík [9] (essentially) considers τ in his proof that Y^* is not homogeneous if Y is not pseudocompact. Let us indicate why we used w rather than τ, and what makes that τ could be used for Y^*.

In the proof of Lemma 4 we used the shyness, implicit in the definition of w, in an essential way (in the proof that $\bar{F} \cap \bar{G} = \emptyset$). Indeed, Lemma 4 is false for τ. For if X is the product of 2^ω discrete 2-point spaces, then X is a homogeneous space, which includes a copy of $\beta\omega$, so $\tau(x, X) = \beta\omega - \omega$ (but $w(x, X) = \emptyset$) for all $x \in X$.

Now, let Y be non-pseudocompact. We prove that Y^* is not homogeneous by observing that Lemma 3 holds for τ instead of w. Since Y^* includes a copy of $\beta\omega$ (cf. Section 4), it suffices to show that $\tau(x, Y^*) \neq \beta\omega - \omega$ for some $x \in Y^*$ (note the difference with Lemma 4). There is an embedding $g\colon \beta\omega \to \beta Y$ with $g^{\to}\omega$, a C-embedded subset of Y, and (hence) $g^{\to}(\beta\omega - \omega) \subseteq Y^*$. Let $q \in \beta\omega - \omega$ and let $x = g(q)$. Suppose that $p \in \tau(x, Y^*)$. Then $x = f(p)$ for some embedding $f\colon \beta\omega \to Y^*$. Since $g^{\to}\omega$ is shy in βY, being C-embedded in Y (see [9], p. 706), the same argument as above shows that $p \sqsubseteq q$. It follows from (a) and (e) of Lemma 5 that $|\tau(x, Y^*)| \leqslant 2^\omega$, whence $\tau(x, Y^*) \neq \beta\omega - \omega$, as required. This is, of course, Frolík's argument [9].

Since $w(x, X) \subseteq \tau(x, Y)$ for all X and $x \in X$, this suggests the question of whether we can improve Lemma 4, and have $|w(x, X)| \leqslant 2^\omega$ for all $x \in X$ (or for at least one $x \in X$). This appears difficult since we do not get an upper bound for $w(x, X)$ for free, as in the argument just indicated. This question is of interest, since a positive answer would eliminate the need for (d) of Lemma 5. (**P 1070**)

4. βX for non-pseudocompact X. Here we prove Lemma 2.

Let $^-$ be the closure operator in βX.

Since X is not pseudocompact, there is a non-empty closed G_δ-subset G of βX which misses X. Let $Y = \beta X - G$. Then $X \subseteq Y \subseteq \beta X$, whence $\beta Y = \beta X$.

Y is σ-compact, hence realcompact, but not compact. Consequently, $|\beta Y - Y| = 2^{2^\omega}$ (see [11], 9.12). Thus we can find a countably infinite, discrete subset S of βY with $S \subseteq G = \beta Y - Y$. We claim that S is as required. Clearly, $\bar{S} \subseteq G \subseteq \beta X - X$.

We first show that S is C^*-embedded. Clearly, S is closed in $Y \cup S$. But $Y \cup S$ is normal, being σ-compact, hence S is C^*-embedded in $\beta(Y \cup S) = \beta Y = \beta X$.

We next show that S is a shy subset of βX. Let $A \subseteq S$, and let B be any countable subset of βX such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Then A and B have disjoint closures in $Y \cup A \cup B$. But $Y \cup A \cup B$ is normal, being σ-compact, so A and B have disjoint closures in $\beta(Y \cup A \cup B) = \beta Y = \beta X$.

5. Example. *There is a pseudocompact homogeneous space, no compactification of which is homogeneous.*

Give the countable ordinals ω_1 the usual topology, let K be the Cantor discontinuum 2^ω, and let $L = K\times\omega_1$. Define a family $\mathscr{B}$ of compact open subsets of L by

$$\mathscr{B} = \{K\times[0, a]:\ a\in\omega_1\},$$

and note that

(4) for every countable $C\subseteq L$ there is a $B\in\mathscr{B}$ with $C\subseteq B$.

This clearly implies that L is pseudocompact, in fact – even countably compact. It follows also that L is homogeneous. Indeed, just observe that every member of $\mathscr{B}$ is clopen in L and, being a compact zero-dimensional metrizable space without isolated points, is homeomorphic to K (see [13]).

Finally, let bL be any compactification of L. By (4), no sequence in L converges to a point of $bL-L$. Hence no point of $bL-L$ is a countable neighborhood base. But, clearly, every point of L has a countable neighborhood base. Hence bL is not homogeneous.

One can also consider $\omega_1\times K$, equipped with the topology which induces the lexicographic order on $\omega_1\times K$.

6. Questions. 1. If X is a compact topological group, is X supercompact? (Added in proof: Answered affirmatively by Mills [17].)

As noted in the Introduction, compact topological groups are dyadic, hence they are the continuous image of a supercompact Hausdorff space. It is unknown if a dyadic space must be supercompact [7], but a continuous Hausdorff image of a supercompact Hausdorff space need not be supercompact [18].

2. If Y is as in Theorem 2, is it true that no power of Y is homogeneous? (**P 1071**)

Partial answers are given in [5], where it is shown, among others, that no power of $\beta\omega$ or of $\beta\omega-\omega$ is homogeneous.

REFERENCES

[1] M. G. Bell and J. van Mill, *The compactness number of a topological space* (to appear).

[2] W. W. Comfort, *Ultrafilters: Some old and some new results*, Bulletin of the American Mathematical Society 83 (1977), p. 417-455.

[3] – and S. Negrepontis, *The theory of ultrafilters*, Berlin 1974.

[4] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific Journal of Mathematics 16 (1966), p. 483-496.

[5] E. K. van Douwen; *Nonhomogeneity of products of preimages and π-weight*, Proceedings of the American Mathematical Society 69 (1978), p. 183-192.

[6] – *Remote points*, Dissertationes Mathematicae (to appear).

[7] – and J. van Mill, *Supercompact spaces*, Topology and its Applications (to appear).

[8] R. Engelking and A. Pełczyński, *Remarks on dyadic spaces*, Colloquium Mathematicum 11 (1963), p. 55-63.

[9] Z. Frolík, *Nonhomogeneity of $\beta P - P$*, Commentationes Mathematicae Universitatis Carolinae 8 (1967), p. 705-709.

[10] — *Homogeneity problems for extremally disconnected spaces*, ibidem 8 (1967), p. 757-763.

[11] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, N. J., 1960.

[12] I. Glicksberg, *Stone-Čech compactifications of products*, Transactions of the American Mathematical Society 90 (1959), p. 369-382.

[13] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1941 (reprinted by Chelsea, New York 1949).

[14] K. Kunen, *Ultrafilters and independent sets*, Transactions of the American Mathematical Society 172 (1972), p. 299-306.

[15] В. Кузьминов, *О гипотезе П. С. Александрова в теории топологических групп*, Доклады Академии наук СССР 125 (1959), p. 727-729.

[16] J. de Vries, *Pseudocompactness and the Stone-Čech compactification for topological groups*, Nieuw Archief voor Wiskunde 23 (3) (1975), p. 35-48.

[17] C. F. Mills, *Compact topological groups are supercompact*, Fundamenta Mathematicae (to appear).

[18] — and J. van Mill, *A nonsupercompact image of a supercompact space*, Houston Journal of Mathematics 5 (1979), p. 241-247.

INSTITUTE FOR MEDICINE AND MATHEMATICS
OHIO UNIVERSITY, ATHENS

Reçu par la Rédaction le 28. 5. 1977