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A GAME OF FAIR DIVISION IN THE NORMAL FORM

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1. Introduction. In this paper we show that the classical game of fair division can be represented by some game in the normal (strategic) form. We prove the existence of the Nash equilibrium point of the game in pure strategies which coincides with the optimal partition in the sense of Urbanik [5].

Let X be an object (e.g., a cake) which has to be divided among n participants (players). The players of the set I are numbered from 1 to n. An ordered partition $P = (A_1, A_2, \ldots, A_n)$ of X is considered to be a *fair division* if each player i receiving the part A_i is "satisfied".

A simple and well-known method of the performing a fair division for two players is "for one to cut, the other to choose". In this method each person can ensure himself that he receives at least a half of the cake according to his own evaluation, independently of what the other has done.

H. Steinhaus in 1944 (cf. Knaster [3]) asked whether a fair procedure could be found for dividing a cake among n participants for $n > 2$. He found a solution for $n = 3$, and S. Banach and B. Knaster (cf. [3]) showed that the solution for $n = 2$ can be extended to arbitrary n. Their division of the cake into n pieces is such that the i-th piece is worth at least $(1/n)$-th of the cake according to the individual measure of the i-th player.

The problem of fair division is convenient to be considered in the language of measure theory.

Let $(X, \mathscr{B}_X)$ be a measurable space, where X is the set which has to be divided among n players, and $\mathscr{B}_X$ is a σ-algebra of subsets of X. Let $\mathscr{P}$ be the set of all ordered measurable partitions $P = (A_1, A_2, \ldots, A_n)$ of X and let μ_i, $i \in I$, be non-atomic probability measures defined on $(X, \mathscr{B}_X)$. Each μ_i represents the individual evaluation of the sets from $\mathscr{B}_X$.

With every partition $P = (A_1, A_2, \ldots, A_n) \in \mathscr{P}$ of X we associate the $(m \times n)$-matrix of real numbers $M(P) = [\mu_j(A_i)]$, $i, j = 1, 2, \ldots, n$. Dvoretzky et al. [2] proved that $M(\mathscr{P})$ (the range of the matrix-valued function M) is convex and compact in $\boldsymbol{R}^{n^2}$ (cf. also [1]).

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative numbers with

$$\sum_{i=1}^{n} \alpha_i = 1.$$

We are interested in giving to each person i a part A_i of X such that $\mu_i(A_i) \geqslant \alpha_i$. Dubins and Spanier [1] derived from the result of Dvoretzky et al. [2] the existence of the fair division, even with a stronger property:

THEOREM 1. *There exists a measurable partition*

$$P = (A_1, A_2, \ldots, A_n)$$

of X such that $\mu_i(A_j) = \alpha_j$ for all $i, j = 1, 2, \ldots, n$.

Urbanik [5], using other methods, obtained the same result earlier. He also proved that if at least two of the measures μ_i, $i \in I$, are different, then there exists a partition $P = (A_1, A_2, \ldots, A_n)$ such that $\mu_i(A_i) > 1/n$ (We put here $\alpha_i = 1/n$, $i \in I$.) Moreover, he showed the existence of an optimal fair division

$$P^* = (A_1^*, A_2^*, \ldots, A_n^*)$$

for which

$$\min_{1 \leqslant j \leqslant n} \mu_j(A_j^*) = \max\,[\min_{1 \leqslant i \leqslant n} \mu_i(E_i)],$$

where the maximum is taken over all measurable partitions

$$P = (E_1, E_2, \ldots, E_n) \in \mathscr{P}$$

(The maximum really exists, since $M(\mathscr{P})$ is compact.)

It is impossible to get the optimal fair division in the pragmatic method of B. Knaster and S. Banach when the players choose their optimal strategies. In this paper we construct a game in the normal form which is in a way equivalent to the classical game of fair division. To any vector of pure strategies of the players in the game in the normal form there corresponds some division and, conversely, to any division there corresponds some vector of pure strategies.

2. A game of fair division in the normal form. Let $\bar{\mu}\colon \mathscr{P} \to \boldsymbol{R}^n$ denote the division vector-valued function such that

$$\bar{\mu}(P) = \big(\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)\big), \qquad P \in \mathscr{P}.$$

The convexity and compactness of the range $M(\mathscr{P})$ imply that the pay-off set $\bar{\mu}(\mathscr{P})$ in the game of fair division is also convex and compact.

DEFINITION (cf. [4]). A convex set $K \subset \boldsymbol{R}^n$ has *dimension* k if, for some $x \in K$, $K - x$ generates a linear manifold of dimension k.

It is easy to verify that if there exist two measures μ_i, μ_j, $i, j \in I$, and a set $A \in \mathscr{B}_X$ such that $0 < \mu_i(A) < \mu_j(A)$, then the set $\bar{\mu}(\mathscr{P})$ has dimension n. If $\mu_1 = \mu_2 = \ldots = \mu_n$, then the set $\bar{\mu}(\mathscr{P})$ has dimension $n-1$.

THEOREM 2. *Assume that the pay-off set $\bar{\mu}(\mathscr{P})$ in the game of fair division has dimension n. Then there exists a game*

$$G = (S_1, S_2, \ldots, S_n, f_1, f_2, \ldots, f_n)$$

(where S_i is the set of strategies and f_i is the pay-off function for the i-th player) with the following properties:

(i) *for any vector of pure strategies $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ there is a partition $P \in \mathscr{P}$ such that*

$$\bar{\mu}(P) = (f_1(s_1), f_2(s_2), \ldots, f_n(s_n))$$

and, conversely,

(ii) *for any partition $P \in \mathscr{P}$ there is a vector of pure strategies $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ such that*

$$(f_1(s_1), f_2(s_2), \ldots, f_n(s_n)) = \bar{\mu}(P).$$

Moreover, the sets of strategies S_i are compact and the pay-off functions f_i are continuous, $i = 1, 2, \ldots, n$.

Proof. Our proof is based on the following well-known theorem: any two compact convex sets of the same dimension are homeomorphic (cf. [4]). We put $S_i = [0, 1]$, $i = 1, 2, \ldots, n$. Let

$$A = [0, 1]^n - \frac{1}{n}\mathbf{1} = \left[-\frac{1}{n}, \frac{n-1}{n}\right]^n \subset \mathbf{R}^n$$

and

$$B = \bar{\mu}(\mathscr{P}) - \frac{1}{n}\mathbf{1},$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^n$. Clearly, the origin $\mathbf{0}$ is an interior point of the set $A \cap B$. Define a map $g\colon A \to B$ by

$$g(x) = \frac{p_A(x)}{p_B(x)}x, \qquad x \in A \subset \mathbf{R}^n,$$

where p_A and p_B are the Minkowski functionals for the sets A and B, respectively. The map g is a homeomorphism. We put

$$f(s) = g\left(s - \frac{1}{n}\mathbf{1}\right) + \frac{1}{n}\mathbf{1}, \qquad s \in [0, 1]^n.$$

It is easy to see that f: $[0, 1]^n \to \bar{\mu}(\mathscr{P})$ and that f is also a homeomorphism. Now we put $f_i = \pi_i f$, where π_i is the projection of $\boldsymbol{R}^n$ onto the i-th coordinate. The game

$$G = ([0, 1], \ldots, [0, 1], f_1, f_2, \ldots, f_n)$$

satisfies the properties stated in the assertion of Theorem 2.

THEOREM 3. *Let* $G = ([0, 1], \ldots, [0, 1], f_1, f_2, \ldots, f_n)$ *be the game of fair division in the normal form in the sense of Theorem* 2. *Then the vector*

$$\mathbf{1} = (1, 1, \ldots, 1) \in \boldsymbol{R}^n$$

is the Nash equilibrium point of the game G.

Proof. We have to prove that, for any pure strategy $s_i \in [0, 1]$,

$$f_i(1, 1, \ldots, 1, s_i, 1, \ldots, 1) \leqslant f_i(\mathbf{1}) \quad \text{for all } i \in I. \tag{1}$$

Suppose that there exist $i \in I$ and $s_i \in [0, 1)$ such that (1) is not satisfied. Let $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, $j \in I$, be the basis in $\boldsymbol{R}^n$. We put

$$s = (1, 1, \ldots, 1, s_i, 1, \ldots, 1) = \sum_{j \neq i}^{n} e_j + e_i s_i.$$

Clearly, $e_j \in \bar{\mu}(\mathscr{P})$, $j \in I$. We have

$$f_i(s) > f_i(\mathbf{1}) = \frac{p_A\left(\dfrac{n-1}{n}\mathbf{1}\right)}{p_B\left(\dfrac{n-1}{n}\mathbf{1}\right)} \frac{n-1}{n} + \frac{1}{n}, \tag{2}$$

where

$$p_A\left(\frac{n-1}{n}\mathbf{1}\right) = \inf\left\{t:\ t > 0,\ \frac{1}{t}\left(\frac{n-1}{n}\mathbf{1}\right) \in \left[-\frac{1}{n}, \frac{n-1}{n}\right]^n\right\} = 1,$$

and

$$f_i(s) = \frac{p_A(s - n^{-1}\mathbf{1})}{p_B(s - n^{-1}\mathbf{1})}\left(s_i - \frac{1}{n}\right) + \frac{1}{n},$$

$$p_A\left(s - \frac{1}{n}\mathbf{1}\right) = \inf\left\{t:\ t > 0,\ \frac{1}{t}\left(\frac{n-1}{n}, \ldots, \frac{n-1}{n}; s_i - \frac{1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}\right) \in \left[-\frac{1}{n}, \frac{n-1}{n}\right]^n\right\} = 1.$$

Put

$$a = p_B\left(\frac{n-1}{n}\mathbf{1}\right) \quad \text{and} \quad b = p_B\left(s - \frac{1}{n}\mathbf{1}\right), \quad a, b > 0.$$

It follows from (2) that

$$\frac{1}{a}\frac{n-1}{n} < \frac{1}{b}\left(s_i - \frac{1}{n}\right). \tag{3}$$

Since $s_i < 1$, we have $b < a$. By the convexity of the set $\bar{\mu}(\mathscr{P})$ we get

$$\lambda f(s) + (1-\lambda)e_i \in \bar{\mu}(\mathscr{P}) \quad \text{for any } \lambda \ (0 \leqslant \lambda \leqslant 1).$$

We shall find $\lambda_0 \in (0, 1)$ such that the vector

$$w = \lambda_0 f(s) + (1-\lambda_0)e_i$$

belongs to the line l represented by the parametric equations $z_i = t$, $i \in I$, $t \in \boldsymbol{R}$. It is easily seen that

$$f(\mathit{1}) = \left(\frac{1}{a}\frac{n-1}{n}\mathit{1} + \frac{1}{n}\mathit{1}\right)$$

also belongs to the line l. We have

$$\lambda f(s) + (1-\lambda)e_i = \lambda \frac{1}{b}\sum_{j \neq i}^{n}\left(\frac{n-1}{n} + \frac{b}{n}\right)e_j$$
$$+ \lambda\left(\frac{s_i - 1/n}{b} + \frac{1}{n}\right)e_i + (1-\lambda)e_i.$$

From the equation

$$\frac{\lambda}{b}\left(\frac{n-1}{n} + \frac{b}{n}\right) = \frac{\lambda}{b}\left(s_i - \frac{1-b}{n}\right) + 1 - \lambda \tag{4}$$

we obtain

$$\lambda_0 = \frac{b}{1 - s_i + b} \quad \text{and} \quad \lambda_0 \in (0, 1).$$

By (3) and (4) we have

$$w_j = \frac{\lambda_0}{b}\left(\frac{n-1}{n} + \frac{b}{n}\right) = \frac{\lambda_0}{b}\left(s_i - \frac{1-b}{n}\right) + 1 - \lambda_0$$
$$= \lambda_0\left[\frac{1}{b}\left(s_i - \frac{1}{n}\right) + \frac{1}{n}\right] + 1 - \lambda_0 > \lambda_0\left[\frac{1}{a}\frac{n-1}{n} + \frac{1}{n}\right] + 1 - \lambda_0$$
$$> \frac{1}{a}\frac{n-1}{n} + \frac{1}{n} = f_k(\mathit{1}) \quad \text{for } j, k \in I.$$

Hence $w > f(\mathit{1})$ and

$$w - \frac{1}{n}\mathit{1} > f(\mathit{1}) - \frac{1}{n}\mathit{1}, \quad w - \frac{1}{n}\mathit{1} \in \bar{\mu}(\mathscr{P}) - \frac{1}{n}\mathit{1}, \tag{5}$$

but

$$a = \inf\left\{t \in \boldsymbol{R}\colon\ t > 0,\ \frac{1}{t}\left(\frac{n-1}{n}\mathit{1}\right) \in \bar{\mu}(\mathscr{P}) - \frac{1}{n}\mathit{1}\right\}.$$

Let t_0 be the positive number satisfying the equation

$$w - \frac{1}{n}\mathit{1} = \frac{1}{t_0}\left(\frac{n-1}{n}\mathit{1}\right).$$

It follows from (5) that $t_0 < a$. This inequality contradicts the definition of the number a and completes the proof.

It is easy to verify that the Nash equilibrium $\mathit{1} = (1, \ldots, 1)$ corresponds to the optimal fair division in the sense of Urbanik [5].

REFERENCES

[1] L. E. Dubins and E. H. Spanier, *How to cut a cake fairly*, Amer. Math. Monthly 68 (1961), pp. 1–17.

[2] A. Dvoretzky, A. Wald and J. Wolfowitz, *Relation among certain ranges of vector measures*, Pacific J. Math. 1 (1951), pp. 59–74.

[3] B. Knaster, *Sur le problème du partage pragmatique de H. Steinhaus*, Annales de la Société Polonaise de Mathématique 19 (1946), pp. 228–230.

[4] T. Parthasarathy and T. E. S. Raghavan, *Some Topics in Two-Person Games*, New York 1971.

[5] K. Urbanik, *Quelques théorèmes sur les mesures*, Fund. Math. 41 (1954), pp. 150–162.

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