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MATROIDS AND DUALITY

BY

D. A. HIGGS (WATERLOO, ONTARIO, CANADA)

Whitney defined duality for finite matroids in [8], and in [5] Sierpiński described the same duality, but for the much more extensive class of Fréchet (V)-spaces. In the present note* this duality is used to define matroids which are not necessarily finite, nor even finitary. The resulting definition is closely related (see Section 4) to Minty's elegant self-dual definitions [2] and provides a feasible starting-point for a solution to Rado's Problem P 531 [3], namely, to construct a theory of non-finitary matroids. Within the class of all matroids, two subclasses are distinguished: B-matroids (Section 3) and C-matroids (Section 4). B-matroids, especially, have properties closely analogous to those of finite matroids. Some of the results obtained here are used in characterizing those infinite graphs which give rise to matroids [1].

1. Spaces in general. A *space* (E, ∂) is a set E together with a function ∂ from the set $\mathscr{P}(E)$ of subsets of E to itself such that

(i) $A \subseteq B$ *implies* $\partial A \cap \partial B$, and

(ii) $A \prec B$ *implies* $B \subseteq A \cup \partial A$ *or* $B \cap \partial B \subseteq A$.

($A \prec B$ means that A is covered by B in $\mathscr{P}(E)$ ordered by inclusion.)

It is easily verified that an equivalent definition of a space is obtained if we replace (ii) by the condition: $x \in \partial A$ implies $x \in \partial(A \setminus x)$. If (E, ∂) is a space, let $\bar{\partial} A = A \cup \partial A$ for all $A \subseteq E$; then $\partial A = \{x; x \in \bar{\partial}(A \setminus x)\}$ for all $A \subseteq E$. Schmidt [4] discusses various properties of spaces, particularly those relevant to matroids.

(1) *If* (E, ∂) *is a space, then so is* (E, ∂^*), *where* $\partial^* A = E \setminus \partial(E \setminus A)$ *for all* $A \subseteq E$.

This is evident from the form of conditions (i) and (ii) above. Clearly $\partial^{**} = \partial$. (E, ∂^*) is called the *dual* of (E, ∂) and, for any property P of spaces, (E, ∂) is said to be *dually* P if and only if (E, ∂^*) is P. An alternative

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description of duality is conveniently stated using the notion of a painting [2]. A *painting* of a set E is a triple (A, x, B) where A and B are disjoint subsets of E, x is in $E\setminus(A \cup B)$, and $A \cup x \cup B = E$. Then clearly

(2) *The spaces* (E, ∂_1) *and* (E, ∂_2) *are duals of each other if and only if, for all paintings* (A, x, B) *of* E, *exactly one of the following two possibilities holds*: (i) $x \in \partial_1 A$; (ii) $x \in \partial_2 B$.

(3) *If* (E, ∂) *is a space and* $S \subseteq E$, *then* $(S, \partial.S)$ *is a space, where* $(\partial.S)A = S \cap \partial A$ *for all* $A \subseteq S$.

This is straight forward. $(S, \partial.S)$ is called the *subspace* of (E, ∂) on S. If (E, ∂) is a space and $S \subseteq E$, we write $\partial \times S$ for $(\partial^*.S)^*$ (so that $(\partial \times S)A = S \cap \partial(A \cup (E\setminus S))$ for all $A \subseteq S$) and refer to a space of the form $(T, (\partial.S)\times T)$, where $T \subseteq S \subseteq E$, as a *minor* of (E, ∂). The notations $\partial.S$, $\partial \times S$ and the term "minor" are Tutte's [7]. It may be verified that the results 3.331 through 3.36 of [7] hold for arbitrary spaces.

Let A be a subset of a space (E, ∂). Then A is said to be: *closed* (in (E, ∂)) if $\partial A \subseteq A$; *pithy* if $A \subseteq \partial A$; *dense* if $A \cup \partial A = E$; *discrete* if $A \cap \partial A = \emptyset$; a *base* if A is both dense and discrete; and *open* if $E\setminus A$ is closed. The following results are immediate.

(4) *If* (E, ∂) *is a space and* $A \subseteq E$, *then*

(i) A *is closed in* (E, ∂) *if and only if* $E\setminus A$ *is pithy in* (E, ∂^*);

(ii) A *is dense in* (E, ∂) *if and only if* $E\setminus A$ *is discrete in* (E, ∂^*); *and*

(iii) A *is a base of* (E, ∂) *if and only if* $E\setminus A$ *is a base of* (E, ∂^*) (cf. [5]).

Note. When a given space (E, ∂) is being discussed, we usually omit explicit reference to it provided no ambiguity arises thereby. In particular, we write S for $(S, \partial.S)$; and "closed" for "closed in (E, ∂)" etc. This last usage is especially appropriate for pithyness and discreteness since these are intrinsic properties in the sense that A is pithy in (E, ∂) if and only if A is pithy in $(A, \partial.A)$ — likewise for discreteness.

2. Matroids in general. A space (E, ∂) is said to be *transitive* if $\bar{\partial}$ is idempotent (and hence a closure). We define a *matroid* to be a space which is both transitive and dually transitive. Thus the dual of a matroid is a matroid.

(5) *Let* (E, ∂) *be a space. Then the following conditions are equivalent*:

(i) (E, ∂) *is transitive*,

(ii) $\partial(A \cup \partial A) \subseteq A \cup \partial A$ *for all* $A \subseteq E$,

(iii) $\partial A = \{x$; *every closed set containing* $A\setminus x$ *contains* $x\}$ *for all* $A \subseteq E$.

Proof. (i) and (ii) are clearly equivalent. Suppose that (i) holds and let $A \subseteq E$. Then $\bar{\partial}(A\setminus x)$ is the intersection of the closed sets containing

$A \setminus x$ and since $x \in \partial A$ if and only if $x \in \bar{\partial}(A \setminus x)$, (iii) follows. Now suppose that (iii) is given to hold and that $x \notin A \cup \partial A$. Then there is a closed set C such that $A \subseteq C$, $x \notin C$. But then $\partial A \subseteq C$ and hence $\partial(A \cup \partial A) \subseteq C$, so that $x \notin \partial(A \cup \partial A)$. It follows that (ii) holds, q.e.d.

On dualizing this result and applying (4), we obtain

(6) *Let (E, ∂) be a space. Then the following conditions are equivalent*:

(i) (E, ∂) *is dually transitive,*

(ii) $A \cap \partial A \subseteq \partial(A \cap \partial A)$ *for all* $A \subseteq E$ (1),

(iii) $\partial A = \{x;\ x$ *is in some pithy subset of* $A \cup x\}$ *for all* $A \subseteq E$.

(7) *Every minor of a transitive space is transitive; likewise for dually transitive spaces and for matroids.*

Proof. By virtue of the formulae $(\partial . S)^* = \partial^* \times S$ and $(\partial \times S)^* = \partial^* . S$, it is sufficient to show this for transitive spaces only, and in this case the result has essentially been given by Tarski ([6], Satz I.6), q.e.d.

The following two notions frequently occur in matroid theory. Let (E, ∂) be a space. Then (E, ∂) is said to be:

(i) *finitely transitive* if $x \in \partial(A \cup y)$ and $y \in \partial A$ implies $x \in \partial A$; and

(ii) *exchange* if $x \in \partial(A \cup y)$ implies $x \in \partial A$ or $y \in \partial(A \cup x)$

for all $A \subseteq E$ and distinct $x, y \in E \setminus A$.

It is easily verified that

(8) (i) *A transitive space is finitely transitive.*

(ii) *A space is finitely transitive if and only if it is dually exchange.*

Thus every dually transitive space, and in particular every matroid, is exchange.

3. *B*-**matroids.** Define the properties B_1 and B_2 of a space (E, ∂) as follows:

B_1: *every discrete set is contained in a base*;

B_2: *if A is discrete, D is dense, and $A \subseteq D$, then there is a base B such that $A \subseteq B \subseteq D$.*

Clearly B_2 implies B_1, and B_1 implies that bases exist. We define a *B-matroid* to be a transitive space each of whose subspaces is B_1. That B-matroids are indeed matroids is a consequence of (12) below. The following result shows that those matroids which have been mainly considered hitherto are B-matroids. A space (E, ∂) is said to be *finitary* if, for all $x \in E$ and $A \subseteq E$, $x \in \partial A$ implies $x \in \partial A_1$ for some finite subset A_1 of A.

(9) *Every finitary transitive exchange space is a B-matroid.*

Proof. Let S be a subset of such a space. It follows from the finitary property that every discrete subset of S is contained in a maximal discrete

(1) I am grateful to Dr. D. W. T. Bean for pointing out (ii) to me.

subset of S; and from the exchange property that a maximal discrete subset of S is a base of S, q.e.d.

(10) *A space is a B-matroid if and only if each of its subspaces is* B_2.

Proof. Suppose that (E, ∂) is a B-matroid and let $S \subseteq E$. Let $A \subseteq D$ be such that A and D are respectively discrete and dense in S. Then, since D is B_1, there is a base B of D such that $A \subseteq B$. From $D \subseteq \bar{\partial} B$ we have $S \subseteq \bar{\partial} D \subseteq \bar{\partial}^2 B = \bar{\partial} B$ and thus B is a base of S. This shows that S is B_2. Now suppose that (E, ∂) is a space in which each subspace is B_2. We have to show that (E, ∂) is transitive. Let $X \subseteq E$ and let B be a base of X. X is dense in $\bar{\partial} X$ and therefore there is a base B' of $\bar{\partial} X$ such that $B \subseteq B' \subseteq X$. Clearly we must have $B' = B$, so that B is a base of $\bar{\partial} X$. By the same argument, B is a base of $\bar{\partial}^2 X$, from which it follows that $\bar{\partial}^2 X = \bar{\partial} X$, q.e.d.

(11) *Let (E, ∂) be a B-matroid and let $A \subseteq S \subseteq E$. Then*

(a) *the following conditions are equivalent*:

(i) *A is dense in $(S, \partial \times S)$*,

(ii) *$A \cup B$ is dense in E for all bases B of $E \setminus S$*,

(iii) *$A \cup B$ is dense in E for some base B of $E \setminus S$*;

(b) *the same equivalences hold with "dense" replaced by "discrete"; and*

(c) *the same equivalences hold with "dense in" replaced by "a base of"* (cf. [7], 3.53).

Proof. Since $E \setminus S$ has a base, (ii) trivially implies (iii) in each case.

(a) Here (i) holds if and only if $S \subseteq A \cup \partial(A \cup (E \setminus S))$. Suppose that this is so and let B be a base of $E \setminus S$. Then $\bar{\partial}(A \cup B) = \bar{\partial}(A \cup (E \setminus S))$ by virtue of transitivity, from which it follows that $A \cup B$ is dense in E. This shows that (i) implies (ii). Now suppose that (iii) holds. Then $S = S \cap E = S \cap (A \cup B \cup \partial(A \cup B)) = A \cup (S \cap \partial(A \cup B)) \subseteq A \cup \partial(A \cup \cup (E \setminus S))$. Hence (iii) implies (i), and (a) is proved.

(b) This time (i) holds if and only if $A \cap \partial(A \cup (E \setminus S)) = \emptyset$. Suppose that this is so and let B be a base of $E \setminus S$. Then B is contained in some base C of $A \cup B$. Since $A \cap \partial(A \cup B) \subseteq A \cap \partial(A \cup (E \setminus S))$, we must have $C = A \cup B$. It follows that (i) implies (ii). Now suppose that (iii) holds and that $x \in A \cap \partial(A \cup (E \setminus S))$. Then $x \in \bar{\partial}((A \setminus x) \cup (E \setminus S)) = \bar{\partial}((A \setminus x) \cup B)$ by virtue of transitivity and therefore $x \in \partial(A \cup B)$, contrary to the discreteness of $A \cup B$. Hence $A \cap \partial(A \cup (E \setminus S)) = \emptyset$. This shows that (iii) implies (i), and (b) is proved.

(c) This is an immediate consequence of (a) and (b), q.e.d.

(12) *The dual of a B-matroid is a B-matroid.*

Proof. Let (E, ∂) be a B-matroid. By (10), the result will be obtained if we show that each subspace of (E, ∂^*) is B_2. Let $A \subseteq D \subseteq S \subseteq E$ be such that A and D are respectively discrete and dense in $(S, \partial^*.S)$. Then by (4), $S \setminus A$ and $S \setminus D$ are respectively dense and discrete in $(S, \partial \times S)$. Let X be a base of $E \setminus S$. By (11), $(S \setminus A) \cup X$ and $(S \setminus D) \cup X$ are respectively dense and discrete in E. Hence there exists a base Y of E such that $(S \setminus D) \cup X \subseteq Y \subseteq (S \setminus A) \cup X$. Now clearly $(Y \cap S) \cup X = Y$. Therefore by (11) again, $Y \cap S$ is a base of $(S, \partial \times S)$. We conclude from (4) that $B = S \setminus (Y \cap S)$ is a base of $(S, \partial^*.S)$. Since $A \subseteq B \subseteq D$, this proves the result, q.e.d.

(13) *Every minor of a B-matroid is a B-matroid.*

Proof. For subspaces this is immediate from (7) and the definition of a B-matroid. The result for arbitrary minors follows from this case by virtue of (12) and the formula $(\partial \times S)^* = \partial^*.S$, q.e.d.

4. C-matroids. A transitive space (E, ∂) is said to be *coatomic* if each closed subset of E is the intersection of maximal proper closed subsets of E, that is, if each open subset of E is the union of minimal non-empty open subsets of E. We define a *C-matroid* to be a matroid which is both coatomic and dually coatomic. Thus the dual of a C-matroid is a C-matroid.

We now relate the present notions to those of Minty [2]. If $\mathscr{C}$ is any set of subsets of a set E, then there is a unique transitive space (E, ∂) on E whose open sets are the unions of members of $\mathscr{C}$ — we say that $\mathscr{C}$ *open-generates* (E, ∂). The following result is an easy consequence of (2) and (5):

(14) *Let* $\mathscr{C}, \mathscr{D} \subseteq \mathscr{P}(E)$. *Then* $\mathscr{C}$ *and* $\mathscr{D}$ *open-generate a matroid on* E *and its dual respectively if and only if* (G-I-II): *for all paintings* (A, x, B) *of* E, *exactly one of the following two possibilities holds*: (i) *there exists* $C \in \mathscr{C}$ *such that* $x \in C \subseteq A \cup x$; (ii) *there exists* $D \in \mathscr{D}$ *such that* $x \in D \subseteq B \cup x$.

Condition (G-I-II) is equivalent to the conjunction of the conditions (G-I) and (G-II) of [2]. It follows that Minty's notion of a pregraphoid — with the finiteness condition removed — is equivalent to that of a general matroid in which sets $\mathscr{C}$ and $\mathscr{D}$ of non-empty open-generators for the matroid and its dual respectively have been distinguished. The next result is a direct consequence of (14).

(15) *Let* $\emptyset \notin \mathscr{C}, \mathscr{D} \subseteq \mathscr{P}(E)$. *Then* $\mathscr{C}$ *and* $\mathscr{D}$ *consist of the minimal non-empty open sets of a C-matroid on E and its dual respectively if and only if* (G-I-II) *and* (G-III): *no member of* $\mathscr{C}$ *contains another properly*; *no member of* $\mathscr{D}$ *contains another properly.*

Condition (G-III) here is the same as that of [2]. Thus Minty's notion of a graphoid — again with the finiteness condition removed — is equivalent to that of a C-matroid.

(16) *Every B-matroid is a C-matroid.*

Proof. Let (E, ∂) be a B-matroid. It is sufficient to prove that (E, ∂) is coatomic. Let A be a closed subset of E and let $x \in E \setminus A$. Let B be a base of A. Then $B \cup x$ is discrete, as may be seen using the fact that (E, ∂) is exchange. Hence E has a base of the form $B \cup x \cup C$, where $(B \cup x) \cap C = \emptyset$. Again using exchange, it may be verified that $\bar{\partial}(B \cup C)$ is a maximal proper closed subset of E which contains A but not x. The result follows, q.e.d.

I do not know whether the converse of this result is true, nor even whether every subspace of a C-matroid is a C-matroid (**P 668**).

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