COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

MULTILINEAR FORMS IN PARETO-LIKE RANDOM VARIABLES AND PRODUCT RANDOM MEASURES

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1. Introduction. This paper deals with the non-L^2 theory of multilinear random forms and with product random measures. Multilinear random forms appear naturally in statistics (especially quadratic forms) and are discrete analogues of multiple stochastic integrals of the form

$$\int \ldots \int f(t_1, \ldots, t_n) dM(t_1) \cdot \ldots \cdot dM(t_n) \tag{1.1}$$

where f is a deterministic function of n variables and M is a stochastic process.

For M being the Brownian notion such integrals were first introduced and studied by K. Itô [10]. The importance of the integrals (1.1) stems from a number of applications such as the representation of non-linear functionals on stochastic processes ([5], [12]), limit theorems for von Mises statistics ([6], [4]), construction of self-similar processes ([21], [19]), and expansions for kernels of solutions of the Schrödinger equation ([7]). A general L^2 theory of multiple integrals was recently developed by Engel [5], but very new results are available for the non-L^2 case (cf. [19], [15], [16]).

As far as multilinear random forms are concerned there exists a fairly complete L^2-theory for quadratic forms ([18], [23]), some stability results for non-L^2 quadratic forms ([20], [24]) and a complete description of a.s. convergent p-stable quadratic forms and their relation to θ_p-radonifying operators ([3]). The results presented in Section 3 deal with multilinear random forms which have tail distributions similar to those of stable random variables. The study of such forms requires a detailed analysis of the tail behavior of distributions of products of random variables (Section 2).

In Section 4 we prove the existence of general product random measures. The quoted above Engel's work marks the first step toward the

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L^2-theory of products of different random measures and our Theorem 4.1 generalizes his results in the case of identical factors. The Theorem 4.1 is the basic step towards the construction of multiple integrals and permits explicit evaluation of the product measure of n-dimensional tetrahedrons. In light of Section 4 the results of Section 3 can be also interpreted as statements about sufficient conditions for the existence of multiple integrals (1.1) of functions that are constant on "rectangles" (cf. [3]).

At last, we would like to mention that simpler versions of the results of this paper were announced, without complete proofs, in [15].

2. Distributional properties of products of Pareto-like random variables. A non-negative random variable X is said to have a *Pareto-like* distribution of order p, $0 < p < \infty$, if

$$\lim_{x\to\infty} x^p P(X > x)$$

exists and is positive. As examples of Pareto-like distributions we quote the Pareto distributions themselves with densities $px^{-(1+p)}$, $x \geqslant 1$, $0 < p < \infty$, the distributions of absolute values of p-stable random variables, $0 < p < 2$, and, more generally, the distributions of absolute values of random variables in the normal domain of attraction of p-stable distributions. Let us note that a Pareto-like distribution of order p has all the moments of order $q < p$ finite and all the moments of order $q \geqslant p$ infinite.

Since the density of Pareto distribution is given explicitly, one can easily evaluate density of the distribution of the product of k i.i.d. Pareto random variables $X_1, \ldots, X_k$:

(2.1)
$$f_{X_1\cdot\ldots\cdot X_k}(x) = \int_{-\infty}^{\infty} \frac{1}{|w|} f_{X_k}(w)\cdot f_{X_1\cdot\ldots\cdot X_{k-1}}\left(\frac{x}{w}\right)dw = \frac{p^k \log^{k-1} x}{(k-1)!\, x^{p+1}}, \qquad x \geqslant 1.$$

For other Pareto-like distributions a computation like (2.1) is not always possible. For general stable distributions, which constitute an important example, we don't even have explicit formulas for densities available. We are able, however, to obtain the asymptotic tail behavior of the products.

THEOREM 2.1. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with Pareto-like distribution such that*

$$\lim_{x\to\infty} \frac{P(X > x)}{x^{-p}} = c. \tag{2.2}$$

Then, for each $k = 1, 2, \ldots$

$$\lim_{x\to\infty} \frac{P(X_1\cdot\ldots\cdot X_k > x)}{x^{-p}(p\log x)^{k-1}} = \frac{c^k}{(k-1)!}. \tag{2.3}$$

Proof. The proof proceeds by induction. The formula (2.3) for $k = 1$ is the same as the assumption (2.2). Let $G_k(x) = P(X_1 \cdot \ldots \cdot X_k > x)$, $F(x) = P(X \leqslant x) = 1 - G_1(x)$ and let $\varphi_k(x) = x^{-p}(p\log x)^{k-1}$.

Let $\varepsilon > 0$. Choose $\alpha > 1$ such that for $u \geqslant \alpha$

$$(1-\varepsilon)cu^{-p} \leqslant G_1(u) \leqslant (1+\varepsilon)cu^{-p}, \tag{2.4}$$

and, by the inductive assumption, such that

$$(1-\varepsilon)\frac{c^k}{(k-1)!}\varphi_k(u) \leqslant G_k(u) \leqslant (1+\varepsilon)\frac{c^k}{(k-1)!}\varphi_k(u). \tag{2.5}$$

To estimate $G_{k+1}(x)$ we decompose it as follows:

$$G_{k+1}(x) = \int_0^{\alpha} G_k(xy^{-1})dF(y) + \int_{\alpha}^{\alpha^{-1}x} G_k(xy^{-1})dF(y) + \int_{\alpha^{-1}x}^{\infty} G_k(xy^{-1})dF(y)$$
$$= I_1(x) + I_2(x) + I_3(x),$$

where $x > \alpha^2$. Observe that

$$I_1(x) \leqslant \int_0^{\alpha} G_k(x\alpha^{-1})dF(y) \leqslant G_k(x\alpha^{-1})$$

and

$$I_3(x) \leqslant \int_{\alpha^{-1}x}^{\infty} dF(y) = G_1(\alpha^{-1}x).$$

Hence $I_1(x) + I_3(x) = o(\varphi_{k+1}(x))$ as $x \to \infty$, and from this point on we need to be concerned only about the behavior of $I_2(x)$.

The inequalities (2.5) yield the following two-sided estimate for I_2:

$$(1-\varepsilon)\frac{c^k}{(k-1)!}\int_{\alpha}^{\alpha^{-1}x}\varphi_k(xy^{-1})dF(y) \leqslant I_2(x) \leqslant (1+\varepsilon)\frac{c^k}{(k-1)!}\int_{\alpha}^{\alpha^{-1}x}\varphi_k(xy^{-1})dF(y).$$

On the other hand

$$\int_{\alpha}^{\alpha^{-1}x}\varphi_k(xy^{-1})dF(y) = -\int_{\alpha}^{\alpha^{-1}x}\varphi_k(xy^{-1})dG_1(y) \tag{2.6}$$

$$= -\varphi_k(xy^{-1})G_1(y)\Big|_{\alpha}^{\alpha^{-1}x} + \int_{\alpha}^{\alpha^{-1}x} G_1(y)\frac{\partial\varphi_k(xy^{-1})}{\partial y}dy$$

$$= o(\varphi_{k+1}(x)) + p^k x^{-p}\int_{\alpha}^{\alpha^{-1}x} G_1(y)y^{p-1}(\log xy^{-1})^{k-1}dy$$

$$-(k-1)p^{k-1}x^{-p}\int_{\alpha}^{\alpha^{-1}x} G_1(y)y^{p-1}(\log xy^{-1})^{k-2}dy,$$

where the last term vanishes if $k = 1$. By (2.2), the term next to the last can be estimated from above and below as follows:

$$(1-\varepsilon)c \int_{\alpha}^{\alpha^{-1}x} y^{-1}(\log xy^{-1})^{k-1}\,dy \leqslant \int_{\alpha}^{\alpha^{-1}x} G_1(y)\,y^{p-1}(\log xy^{-1})^{k-1}\,dy$$
$$\leqslant (1-\varepsilon)c \int_{\alpha}^{\alpha^{-1}x} y^{-1}(\log xy^{-1})^{k-1}\,dy,$$

where

$$\int_{\alpha}^{\alpha^{-1}x} y^{-1}(\log xy^{-1})^{k-1}\,dy = k^{-1}\left[(\log \alpha^{-1}x)^k - (\log \alpha)^k\right],$$

so that

$$(1-\varepsilon)ck^{-1}\varphi_{k+1}(x) + o\big(\varphi_{k+1}(x)\big) \leqslant p^k x^{-p} \int_{\alpha}^{\alpha^{-1}x} G_1(y)\,y^{p-1}(\log xy^{-1})^{k-1}\,dy$$
$$\leqslant (1+\varepsilon)ck^{-1}\varphi_{k+1}(x) + o\big(\varphi_{k+1}(x)\big).$$

For $k > 1$, the last term in (2.6) is $o\big(\varphi_{k+1}(x)\big)$ as $x \to \infty$, by a similar argument, so that, finally

$$(1-\varepsilon)^2 \frac{c^{k+1}}{k!}\varphi_{k+1}(x) + o\big(\varphi_{k+1}(x)\big) \leqslant I_2(x) \leqslant (1+\varepsilon)^2 \frac{c^{k+1}}{k!}\varphi_{k+1}(x) + o\big(\varphi_{k+1}(x)\big).$$

Remark 2.1. We would like to note that the above method of proof also gives the following implication: if $X, X_1, X_2, \ldots$ are i.i.d. random variables such that

$$\overline{\lim_{x\to\infty}} \frac{P(X > x)}{x^{-p}} \leqslant c < \infty,$$

then, for each $k = 1, 2, \ldots$

$$\overline{\lim_{x\to\infty}} \frac{P(X_1 \cdot \ldots \cdot X_k > x)}{x^{-p}(p\log x)^{k-1}} \leqslant \frac{c^k}{(k-1)!}.$$

A similar result can be obtained for $\underline{\lim}$.

The above theorem permits us to show that the products of p-stable random variables have distributions in the domain of attraction of the p-stable law. The following corollary gives an even stronger result.

Corollary 2.1. *Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with a symmetric distribution ν. Assume that ν is in the normal domain of attraction of a symmetric p-stable distribution μ $(0 < p < 2)$, i.e.*

$$\mathscr{L}\Big(n^{-1/p}\sum_{j=1}^{n} Y_j\Big) \Rightarrow \mu \quad \text{as } n \to \infty.$$

Let $k \in N$. If $Z_1^k, Z_2^k, \ldots$ is a sequence of i.i.d. random variables such that

$$\mathcal{L}(Z_1^{(k)}) = \mathcal{L}(Y_1 \cdot \ldots \cdot Y_k)$$

then

$$\mathcal{L}\Big(a_n^{-1/p} \sum_{j=1}^{n} Z_j^{(k)}\Big) \Rightarrow \mu \quad \text{as } n \to \infty,$$

where

$$a_n = a_n(k) = \frac{c^{k-1}}{(k-1)!} n (\log n)^{k-1},$$

and $c = \lim_{x \to \infty} x^p \nu\{(x, \infty)\} = \lim_{x \to \infty} x^p \mu\{(x, \infty)\}$.

Proof. In view of Corollary 6.18(a) of [1] it suffices to show that

$$\lim_{n \to \infty} nP(|Y_1 \cdot \ldots \cdot Y_k| > a_n^{1/p}) = c.$$

Applying Theorem 2.1 we have that the above limit is equal to

$$\frac{c^k}{(k-1)!} \lim_{n \to \infty} n a_n^{-1} (p \log a_n^{1/p})^{k-1} = c.$$

3. Multilinear forms: convergence and tail behavior. Throughout this section $X_1, X_2, \ldots$ is a sequence of i.i.d. symmetric random variables such that

$$\overline{\lim_{x \to \infty}}\, x^p P(|X_1| > x) \leqslant c < \infty,$$

where $0 < p < 2$. The results of the previous section show that for each $k = 1, 2, \ldots$, there exists a constant $C_{k,p}$ such that for all $x > 0$

$$P(|X_1 \cdot \ldots \cdot X_k| > x) \leqslant C_{k,p} x^{-p} (1 + \log_+^{k-1} x). \tag{3.1}$$

Let $k \in N$. In what follows we study the behavior of the multilinear forms

$$Q_n^{(k)}(f) = \sum_{1 \leqslant i_1, \ldots, i_k \leqslant n} f(i_1, \ldots, i_k) X_{i_1} \cdot \ldots \cdot X_{i_k},$$

where f is a real function defined on N^k, and such that $f(i_1, \ldots, i_k) = 0$ whenever two or more indices coincide.

We say that $Q_1^{(k)}(f), Q_2^{(k)}(f), \ldots$ *converge unconditionally* (a.s., in P, in L^q) if for any $\{-1, +1\}$-valued function $\varepsilon = \varepsilon(i_1, \ldots, i_k)$ on N^k the sequence $Q_1^{(k)}(\varepsilon f), Q_2^{(k)}(\varepsilon f), \ldots$ converges. We would like to remark that the unconditional a.s. convergence defined above is weaker than the a.s. convergence for all permutations of f (cf. Ulyanov's example mentioned in [13]).

THEOREM 3.1. *If*

$$N_p^{(k)}(f) \stackrel{\mathrm{df}}{=} \sum_{i_1,\ldots,i_k} |f(i_1, \ldots, i_k)|^p (1+\log_+^{k-1}|f(i_1, \ldots, i_k)|^{-1}) < \infty$$

then the sequence $Q_1^{(k)}(f), Q_2^{(k)}(f), \ldots$ *converges unconditionally in* L^q *for every* $q < p$ *to a* $Q^{(k)}(f)$ *which, for all* $x > 0$, *satisfies the following inequality*

$$P(|Q^{(k)}(f)| > x) \leqslant D_{k,p} x^{-p}(1+\log_+^{k-1} x) N_p^{(k)}(f), \tag{3.2}$$

where

$$D_{k,p} = 2^{k-1} k! \frac{4-p}{2-p} C_{k,p}.$$

Proof. Let us denote $\varphi(x) = x^{-p}(1+\log_+^{k-1} x)$, $x \geqslant 0$. It is easy to check that φ is submultiplicative with a constant 2^{k-1}, i.e. $\varphi(xy) \leqslant 2^{k-1}\varphi(x)\varphi(y)$ for every $x, y \geqslant 0$.

Define

$$U_n = \max_{1 \leqslant i_1,\ldots,i_k \leqslant n} |f(i_1, \ldots, i_k) X_{i_1}, \ldots, X_{i_k}|.$$

Then, suppressing the superscript (k), we get that

$$\begin{aligned} P(|Q_n| > x) &\leqslant P(|Q_n| > x, U_n > x) + P(|Q_n| > x, U_n \leqslant x) \\ &\leqslant P(U_n > x) + P\Big(\Big|\sum_{1 \leqslant i_1,\ldots,i_k \leqslant n} Y(i_1, \ldots, i_k)\Big| > x\Big) \end{aligned} \tag{3.3}$$

where $Y(i_1, \ldots, i_k)$ is the truncation of $f(i_1, \ldots, i_k) X_{i_1} \cdot \ldots \cdot X_{i_k}$ at level x. From (3.1) we obtain that

$$\begin{aligned} P(U_n > x) &\leqslant \sum_{1 \leqslant i_1,\ldots,i_k \leqslant n} P\big(|f(i_1, \ldots, i_k) X_{i_1} \cdot \ldots \cdot X_{i_k}| > x\big) \\ &\leqslant C_{k,p} \sum \varphi\big(x|f(i_1, \ldots, i_k)|^{-1}\big). \end{aligned} \tag{3.4}$$

To estimate the second term in (3.3) observe that Y's are uncorrelated as long as $i_{\pi(1)} < i_{\pi(2)} < \ldots < i_{\pi(k)}$ for a fixed permutation π (and only the above case needs to be considered since f vanishes on all "diagonal" sets by assumption). To simplify the notation we will work only with π being the identity. Then, by Chebyshev's inequality

$$\begin{aligned} &P\Big(\Big|\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} Y(i_1, \ldots, i_k)\Big| > x\Big) \\ &\qquad \leqslant x^{-2} E\big(\sum Y(i_1, \ldots, i_k)\big)^2 = x^{-2} \sum E Y^2(i_1, \ldots, i_k) \\ &\qquad = 2x^{-2} \sum \int_0^x u P\big(|Y(i_1, \ldots, i_k)| > u\big) du \end{aligned} \tag{3.5}$$

$$\leqslant 2C_{k,p}x^{-2}\sum\int_0^x u\varphi(u|f(i_1,\ldots,i_k)|^{-1})du$$

$$= 2C_{k,p}x^{-2}\sum f^2(i_1,\ldots,i_k)\int_0^{x/|f(i_1,\ldots,i_k)|} v\varphi(v)\,dv$$

$$\leqslant 2C_{k,p}x^{-2}\sum f^2(i_1,\ldots,i_k)[1+\log_+^{k-1}(x|f(i_1,\ldots,i_k)|^{-1})]\int_0^{x/|f(i_1,\ldots,i_k)|} v^{1-p}\,dv$$

$$= \frac{2}{2-p}C_{k,p}\sum\varphi(x|f(i_1,\ldots,i_k)|^{-1}).$$

Now (3.3) together with (3.4) and (3.5) and the submultiplicativity of φ give the inequality (3.2) for finite multilinear forms. This permits us to establish that for each fixed $\{-1,+1\}$-valued function $\varepsilon = \varepsilon(i_1,\ldots,i_k)$ on N^k, $Q_1^{(k)}(\varepsilon f)$, $Q_2^{(k)}(\varepsilon f)$, ... is a Cauchy sequence in probability and in L^q, $q < p$, provided $N_p^{(k)}(f) < \infty$.

4. Product random measure. Let M be an atomless random measure (i.e. measure with values in $L^0(\Omega, \mathcal{F}, P)$) on a Polish space T equipped with the Borel σ-field $\mathcal{B}(T)$. We assume that M is independently scattered, i.e. for any collection of pairwise disjoint $A_1,\ldots,A_n\in\mathcal{B}(T)$ the random variables $M(A_1),\ldots,M(A_n)$ are independent. There exists a positive atomless measure μ which is mutually absolutely continuous with respect to M. We call it a *control measure* of M and an explicit formula for μ can be found in [22].

For $B = A_1\times\ldots\times A_n$ we set

$$M^n(B) = M(A_1)\cdot M(A_2)\cdot\ldots\cdot M(A_n),$$

and extend M^n to a finitely additive (but not independently scattered) random set function on the field spanned by "rectangular" sets. Engel [5] (Thm. 4.5) demonstrated that, under high moment assumptions on M, M^n extends to a countably additive L^2-valued measure (actually he proved the result in a more general contex of product of different random measures, see also [14] for L^1 and L^2 theory of product random measures).

Using the above result we are able to prove the following general

THEOREM 4.1. *M^n extends to a countably additive vector measure with values in $L^0(\Omega, \mathcal{F}, P)$.*

Proof. It is well known (see e.g. [17]) that $M = Q+R$, $i = 1, 2, \ldots, n$, i.e. $M(A) = Q(A)+R(A)$ a.s. for every $A\in\mathcal{B}(T)$, where Q and R are independently scattered random measures, Q has all moments finite, and for every $\omega\in\Omega$, $R(\cdot)(\omega)$ is a signed measure with finite support. By Engel's result quoted above, for each subset $I\subset\{1,\ldots,n\} = N$, there exists an L^2-valued countably additive product measure

$$Q^I = \prod_{i \in I} Q$$

defined on $\mathscr{B}(T^I)$. On the other hand, for each $J \subset N$, and every $\omega \in \Omega$, the product measure

$$R^J(\cdot)(\omega) = \prod_{j \in J} R(\cdot)(\omega)$$

is well defined on $\mathscr{B}(T^J)$ as a measure with finite support.

For each $\omega \in \Omega$ and $B \in \mathscr{B}(T^n)$ define

$$\tilde{M}^n(B)(\omega) = \sum_I \int_{T^{N \setminus I}} Q^I(B^u)(\omega) R^{N \setminus I}(du)(\omega),$$

where the summation extends over all subsets $I \subset N$, and $B^u = \{v \in T^I\colon \exists\, w \in B,\ w|_I = v,\ w|_{N \setminus I} = u\}$ for every $u \in T^{N \setminus I}$. Note that, since for every $\omega \in \Omega$ the signed measure $R^{N \setminus I}(\cdot)(\omega)$ has finite support, the integral in the above formula is well defined, and by the independence of Q_I and $R_{N \setminus I}$ we have

$$\begin{aligned}(4.1)\qquad & P\{\omega \in \Omega\colon \int_{T^{N \setminus I}} Q^I(B^u)(\omega) R^{N \setminus I}(du)(\omega) \in E\} \\ & = P \times P\{(\omega_1, \omega_2) \in \Omega \times \Omega\colon \int_{T^{N \setminus I}} Q^I(B^u)(\omega_1) R^{N \setminus I}(du)(\omega_2) \in E\}\end{aligned}$$

for any $B \in \mathscr{B}(T^N)$ and $E \in \mathscr{B}(\boldsymbol{R})$.

Let $B_1, B_2, \ldots$ be a descending sequence of sets from $\mathscr{B}(T^N)$ such that $\bigcap B_k = \emptyset$. For every $I \subset N$ and $u \in T^{N \setminus I}$

$$\lim_{k \to \infty} Q^I(B_k^u)(\omega_1) = 0$$

in $P(d\omega_1)$. Since, for each $\omega_2 \in \Omega$, $R^{N \setminus I}(\cdot)(\omega_2)$ is a measure with finite support, we have, for each $\omega_2 \in \Omega$, that

$$\lim_{k \to \infty} \int_{T^{N \setminus I}} Q^I(B_k^u)(\omega_1) R^{N \setminus I}(du)(\omega_2) = 0$$

in $P(d\omega_1)$. Now the continuity at $\emptyset$ of $\tilde{M}^n$ follows directly from the Fubini's Theorem (our first encounter with it was through [9] in our real analysis courses) and (4.1). Since $\tilde{M}^n(A_1 \times \ldots \times A_n) = M^n(A_1 \times \ldots \times A_n)$ a.s. for any choice of $A_1, \ldots, A_n \in \mathscr{B}(T)$, $\tilde{M}^n$ is a countably additive extension of M^n.

Let M be an atomless independently scattered random measure on $\mathscr{B}([0, T])$. Then the logarithm of the characteristic function of M can be written in the Lévy's form

$$\log E\exp(itM(A))$$
$$= itv(A) - \tfrac{1}{2}\sigma^2(A)t^2 + \int_{-\infty}^{\infty} (e^{itx} - 1 - txI(|x| \leq 1))\pi(A, dx),$$

where v is a signed measure on $\mathscr{B}([0, T])$, σ^2 is a non-negative finite measure defined on $\mathscr{B}([0, T])$ and $\pi(du, dx)$ is a non-negative measure on $\mathscr{B}([0, T] \times \mathbf{R})$ such that $\pi(A \times (-\varepsilon, \varepsilon)^c) < \infty$ for every A and $\varepsilon > 0$, and $\int_{-1}^{+1} x^2 \pi([0, T], dx) < \infty$. Since, for every integer $k \geq 2$,

$$\int_{[0,T]\times[-1,1]} |x|^k \pi(du, dx) = \int_{-1}^{1} |x|^k \pi([0, T], dx) < \infty,$$

we get that

$$M_k(A) = \int_A (M(dt))^k$$

is a well-defined atomless independently scattered random measure. The variational integral above is defined as

$$\lim_{|P|\to 0} \sum (M[(t_i, t_{i+1}) \cap A])^k$$

in probability where $P = \{t_0, t_1, \ldots, t_m\}$ is a partition of $[0, T]$ and $|P| = \max_i |t_{i+1} - t_i|$ (cf. e.g. [2]). Theorem 4.1 and the above comments permit a straightforward adaptation of the proof of Theorem 6.1 of [5] to obtain the following

COROLLARY 4.1. *Let M be an atomless, independently scattered random measure on $\mathscr{B}([0, T])$. Let $H(n_1, \ldots, n_k)$ be the number of distinct ways of partitioning $\{1, \ldots, n\}$ into subsets of sizes $(n_1, \ldots, n_k)$. Then*

$$\int_{0\leq t_1 < \ldots < t_n \leq T} M(dt_1)\cdot \ldots \cdot M(dt_n)$$
$$= \frac{1}{n!} \sum_{k=1}^{n} \sum_{n_1+\ldots+n_k=n} H(n_1, \ldots, n_k)(-1)^{n-k} \prod_{i=1}^{k} (n_i - 1)! \prod_{i=1}^{k} \int_0^T (M(dt))^{n_i}.$$

EXAMPLE 4.1. (a) If $n = 2$ we have $H(2) = 1$, $H(1, 1) = 1$ and

$$\int_{0\leq t_1 < t_2 \leq T} M(dt_1)M(dt_2) = \tfrac{1}{2}\Big[-\int_0^T (M(dt))^2 + M^2([0, T])\Big],$$

which, in the case when $M([0, t]) = B(t)$ is a Brownian motion, reduces to the well-known formula

$$\int_0^1 B(t)\,dB(t) = \tfrac{1}{2}(B^2(1) - 1).$$

(b) In the case when $M([0, t]) = N(t)$ is a homogeneous Poisson process we obtain that

$$\int_{0 \leqslant t_1 < t_2 \leqslant T} dN(t_1)\,dN(t_2) = \tfrac{1}{2}[-N(T)+N^2(T)].$$

(c) If $M([0, t]) = X_p(t)$ is a symmetric p-stable stationary process with independent increments, then $X_{p/k}(t) = \int_0^t (dX_p(t))^k$ is a stationary (p/k)-stable process with independent increments, symmetric if k is odd, and non-negative for even k's (cf. e.g. [8]). Thus

$$\int_{0 \leqslant t_1 < t_2 \leqslant T} dX_p(t_1)\,dX_p(t_2) = \tfrac{1}{2}[-X_{p/2}(T)+X_p^2(T)],$$

and more generally, the product stable measure of the tetrahedron $\{0 \leqslant t_1 < \ldots < t_n \leqslant T\}$ is a polynomial in stable random variables of orders p/k, $k = 1, 2, \ldots, n$ (cf. [11]).

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Reçu par la Rédaction le 15. 05. 1984