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THE ABSOLUTE HARMONIC SUMMABILITY FACTORS OF INFINITE SERIES

BY

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1. Let $\sum a_n$ be a given infinite series with its n-th partial sum S_n, and let $t_n = t_n^0 = na_n$. By σ_n^a and t_n^a we denote the n-th Cesàro means of order a ($a > -1$) of the sequences $\{S_n\}$ and $\{t_n\}$ respectively. The series $\sum a_n$ is said to be *absolutely summable* (C, a) with index k, or simply *summable* $|C, a|_k$ ($k \geqslant 1$), if [2]

$$\sum n^{k-1} |\sigma_n^a - \sigma_{n-1}^a|^k < \infty. \tag{1.1}$$

Summability $|C, a|_1$ is the same as summability $|C, a|$.

Since

$$t_n^a = n(\sigma_n^a - \sigma_{n-1}^a),$$

condition (1.1) can also be written as

$$\sum \frac{|t_n^a|^k}{n} < \infty. \tag{1.2}$$

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = \sum_{i=0}^{n} p_i, \quad P_{-1} = p_{-1} = 0.$$

The sequence $\{T_n\}$ defined by

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu S_{n-\nu} = \frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu a_{n-\nu} \qquad (P_n \neq 0)$$

defines the Nörlund mean of the sequence $\{S_n\}$ generated by the sequence of constants $\{p_n\}$.

The series $\sum a_n$ is said to be *absolutely summable* (N, p_n) with index k, or *summable* $|N, p_n|_k$, if [1]

$$\sum n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{1.3}$$

When $k = 1$, this definition reduces to the customary definition of absolute Nörlund summability as given by Mears [5].

We shall always write $p_n = 1/(n+1)$ and therefore $P_n \sim \log n$ as $n \to \infty$. In this case, the Nörlund mean reduces to the familiar harmonic mean [6].

A sequence $\{\lambda_n\}$ is said to be *convex* [8], if

$$\Delta^2 \lambda_n \geqslant 0, \quad n = 1, 2, 3, \ldots,$$

where

$$\Delta \lambda_n = \lambda_n - \lambda_{n+1} \quad \text{and} \quad \Delta^2 \lambda_n = \Delta(\Delta \lambda_n).$$

2. McFadden [4] has shown that if a series is absolutely harmonic summable, it is also summable $|C, a|$ for every $a > 0$; but the converse is not always necessarily true. Naturally, now the question arises: if a series $\sum a_n$ is summable $|C, a|$ for certain a, say $a = 1$, then will the additional restrictions in the shape of summability factors $\{\mu_n\}$ ensure the absolute harmonic summability of the factored series $\sum a_n \mu_n$? Applying an answer to this question, Singh [7] has proved the following theorem:

THEOREM S. *If the series $\sum a_n$ is summable $|C, 1|$, then the series $\sum [a_n \log(n+1)]/n$ is summable $|N, 1/(n+1)|$.*

The object of this paper is to extend this result by obtaining the following theorem for summability $|N, 1/(n+1)|_k$:

THEOREM 1. *If the series $\sum a_n$ is summable $|C, 1|_k$ $(k \geqslant 1)$, then the series $\sum [a_n \log(n+1)]/n$ is summable $|N, 1/(n+1)|_k$.*

3. Proof of Theorem 1. Since the case $k = 1$ of the theorem is due to Singh [7], we prove it for $k > 1$ only.

Further, since the series $\sum a_n$ is summable $|C, 1|_k$ $(k \geqslant 1)$, we have

$$\sum \frac{|t_n^1|^k}{n} < \infty. \tag{3.1}$$

Let t_n^* denote the harmonic mean of the series $\sum [a_\nu \log(\nu+1)]/\nu \equiv \sum u_\nu$. Then we have to show that

$$\sum n^{k-1} |t_n^* - t_{n-1}^*|^k < \infty.$$

Now

$$t_n^* = \frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu u_{n-\nu} = u_0 + \frac{1}{P_n} \sum_{\nu=0}^{n-1} P_\nu u_{n-\nu};$$

and, similarly,

$$t_{n-1}^* = u_0 + \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} P_{\nu-1} u_{n-\nu},$$

hence

$$t_n^* - t_{n-1}^* = \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) u_{n-\nu}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n p_{n-\nu} - P_{n-\nu} p_n) \frac{\log(v+1)\cdot a_\nu}{\nu}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{m} (P_n - P_{n-\nu}) p_n \frac{\log(\nu+1)\cdot a_\nu}{\nu} +$$

$$+ \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{m} (p_{n-\nu} - p_n) P_n \frac{\log(\nu+1)\cdot a_\nu}{\nu} +$$

$$+ \frac{1}{P_n P_{n-1}} \sum_{\nu=m+1}^{n} (P_n p_{n-\nu} - P_{n-\nu} p_n) \frac{\log(\nu+1)\cdot a_\nu}{\nu}$$

$$= L_n^1 + L_n^2 + L_n^3,$$

where m is the integral part of $n/2$.

By Minkowski's inequality, it is therefore sufficient to prove that

$$\sum n^{k-1} |L_n^1|^k < \infty, \tag{3.2}$$

$$\sum n^{k-1} |L_n^2|^k < \infty, \tag{3.3}$$

and

$$\sum n^{k-1} |L_n^3|^k < \infty. \tag{3.4}$$

Proof of (3.2). We have

$$\sum n^{k-1} |L_n^1|^k = \sum n^{k-1} \left\{ \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{m} (P_n - P_{n-\nu}) \frac{\log(\nu+1)}{\nu^2} \cdot \nu a_\nu \right| \right\}^k$$

$$\leqslant A\,(^1) \sum n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{\nu=1}^{m-1} \nu |t_\nu^1| \cdot \left| \Delta \left(\frac{(P_n - P_{n-\nu}) \log(\nu+1)}{\nu^2} \right) \right| + \right.$$

$$\left. + |t_m^1| \frac{(P_n - P_{n-m})}{m} \log(m+1) \right\}^k$$

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[1] A is a positive finite constant but is not necessarily the same at each occurrence.

$$\leqslant A\cdot\sum n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\log(\nu+1)}{\nu}\,p_{n-\nu}+\right.$$

$$\left.+\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\log(\nu+1)}{\nu^2}+\frac{|t_m^1|}{m}\log(m+1)\right\}^k$$

$$\leqslant A\cdot\left[\left\{\sum n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\log(\nu+1)}{\nu^2}\right)^k\right\}^{1/k}+\right.$$

$$\left.+\left\{\sum n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\frac{|t_m^1|}{m}\log(m+1)\right)^k\right\}^{1/k}\right]^k$$

$$= A\cdot[X_1^{1/k}+X_2^{1/k}]^k,$$

since

$$P_n-P_{n-\nu}<A$$

and

$$\frac{1}{(n-\nu)}<\frac{1}{\nu}\quad\text{for } 1\leqslant\nu\leqslant m-1.$$

Now,

$$X_1\leqslant A\cdot\sum n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\cdot(\log m)^k\cdot\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|}{\nu^2}\right)^k$$

$$\leqslant A\cdot\sum\frac{1}{n(\log n)^k}\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|^k}{\nu^2}\right)\left(\sum_{\nu=1}^{m-1}\frac{1}{\nu^2}\right)^{k-1}$$

$$\leqslant A\cdot\sum\frac{1}{n(\log n)^k}<A,$$

where the last inequality follows by (3.1).

Further,

$$X_2\leqslant A\cdot\sum\frac{|t_m^1|^k}{m}<A,$$

where again the last inequality follows by (3.1).

This proves (3.2).

Proof of (3.3). We have

$$\sum n^{k-1}|L_n^2|^k=\sum n^{k-1}\left\{\frac{1}{P_{n-1}}\left|\sum_{\nu=1}^{m}\frac{\nu a_\nu(p_{n-\nu}-p_n)\log(\nu+1)}{\nu^2}\right|\right\}^k$$

$$\leqslant A\cdot\sum\frac{1}{n(P_{n-1})^k}\left\{\left|\sum_{\nu=1}^{m}\nu a_\nu\frac{\log(\nu+1)}{\nu(n-\nu+1)}\right|\right\}^k$$

$$\leqslant A\cdot\sum\frac{1}{n(P_{n-1})^k}\left\{\sum_{\nu=1}^{m-1}\nu|t_\nu^1|\cdot\left|\Delta\left(\frac{\log(\nu+1)}{\nu(n-\nu+1)}\right)\right|+\frac{|t_m^1|\cdot\log(m+1)}{(n-m+1)}\right\}^k$$

$$\leqslant A\cdot\sum\frac{1}{n(P_{n-1})^k}\left\{\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\cdot\log(\nu+1)}{\nu(n-\nu+1)}+\right.$$

$$\left.+\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\cdot\log(\nu+1)}{(n-\nu)(n-\nu+1)}+\frac{|t_m^1|\log(m+1)}{(n-m+1)}\right\}^k$$

$$\leqslant A\cdot\left[\left\{\sum\frac{1}{n(P_{n-1})^k}\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|\log(\nu+1)}{\nu(n-\nu+1)}\right)^k\right\}^{1/k}+\right.$$

$$\left.+\left\{\sum\frac{1}{n(P_{n-1})^k}\left(\frac{|t_m^1|\cdot\log(m+1)}{(n-m+1)}\right)^k\right\}^{1/k}\right]^k$$

$$= A\cdot[Y_1^{1/k}+Y_2^{1/k}]^k,$$

since

$$\frac{1}{(n-\nu)}<\frac{1}{\nu}\quad \text{for } 1\leqslant\nu\leqslant m-1.$$

Now,

$$Y_1\leqslant A\cdot\sum\frac{1}{n(P_{n-1})^k}\frac{(\log m)^k}{(n-m+2)^{k/2}}\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|}{\nu(n-\nu+1)^{1/2}}\right)^k$$

$$\leqslant A\cdot\sum\frac{1}{m^{1+k/2}}\left(\sum_{\nu=1}^{m-1}\frac{|t_\nu^1|^k}{\nu^{3/2}}\right)\left(\sum_{\nu=1}^{m-1}\frac{1}{\nu^{3/2}}\right)^{k-1}$$

$$\leqslant A\cdot\sum\frac{1}{m^{1+k/2}}<A$$

by virtue of (3.1) and by

$$\frac{1}{(n-\nu)}<\frac{1}{\nu}\quad \text{for } 1\leqslant\nu\leqslant m-1.$$

Next,

$$Y_2\leqslant A\cdot\sum\frac{|t_m^1|^k}{m}<A$$

by (3.1).

This proves (3.3).

Proof of (3.4). We have

$$\sum n^{k-1}|L_n^3|^k$$

$$= \sum n^{k-1}\left\{\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=m+1}^{n}(P_np_{n-\nu}-P_{n-\nu}p_n)\frac{\log(\nu+1)}{\nu^2}\,\nu a_\nu\right|\right\}^k$$

$$\leqslant A\cdot\sum\frac{n^{k-1}}{(P_nP_{n-1})^k}\left\{\sum_{\nu=m+1}^{n-1}\nu|t_\nu^1|\cdot\left|\Delta\left((P_np_{n-\nu}-P_{n-\nu}p_n)\cdot\frac{\log(\nu+1)}{\nu^2}\right)\right|+\right.$$

$$\left.+|t_n^1|\,\frac{\log(n+1)\cdot P_n}{n}+|t_m^1|\,\frac{\log(m+1)\cdot P_np_{n-m}}{m}\right\}^k$$

$$\leqslant A\cdot\sum\frac{n^{k-1}}{(P_nP_{n-1})^k}\left\{\sum_{\nu=m+1}^{n-1}\frac{|t_\nu^1|\cdot\log(\nu+1)}{\nu^2}\,P_np_{n-\nu}+\right.$$

$$\left.+\sum_{\nu=m+1}^{n-1}\frac{|t_\nu^1|\cdot\log(\nu+1)}{\nu}\,\frac{P_n}{(n-\nu)(n-\nu+1)}+\frac{|t_n^1|\cdot P_nP_{n-1}}{n}\right\}^k$$

$$\leqslant A\cdot\left[\left\{\sum\frac{n^{k-1}}{(P_nP_{n-1})^k}\left(\sum_{\nu=m+1}^{n-1}\frac{|t_\nu^1|\cdot\log(\nu+1)}{\nu}\,\frac{P_n}{(n-\nu)(n-\nu+1)}\right)^k\right\}^{1/k}\right.$$

$$\left.+\left\{\sum\frac{n^{k-1}}{(P_nP_{n-1})^k}\left(\frac{P_nP_{n-1}}{n}\,|t_n^1|\right)^k\right\}^{1/k}\right]^k$$

$$= A\cdot[Z_1^{1/k}+Z_2^{1/k}]^k,$$

since

$$\frac{1}{(n-\nu)}>\frac{1}{\nu}\quad\text{for } m+1\leqslant\nu\leqslant n-1.$$

Now,

$$Z_1\leqslant A\cdot\sum_{n=1}^{M}\frac{1}{n}\left(\sum_{\nu=m+1}^{n-1}\frac{|t_\nu^1|}{(n-\nu)(n-\nu+1)}\right)^k$$

$$\leqslant A\cdot\sum_{n=1}^{M}\frac{1}{n}\left(\sum_{\nu=m+1}^{n-1}\frac{|t_\nu^1|^k}{(n-\nu)(n-\nu+1)}\right)\left(\sum_{\nu=m+1}^{n-1}\frac{1}{(n-\nu)(n-\nu+1)}\right)^{k-1}$$

$$\leqslant A\cdot\sum_{n=1}^{M}\frac{1}{n}\sum_{\nu=1}^{n-1}\frac{|t_\nu^1|^k}{(n-\nu)^2}$$

$$\leqslant A\cdot\sum_{\nu=1}^{M-1}\frac{|t_\nu^1|^k}{\nu}\sum_{n=\nu+1}^{M}\frac{1}{(n-\nu)^2}$$

$$\leqslant A\cdot\sum_{\nu=1}^{M-1}\frac{|t_\nu^1|^k}{\nu}<A$$

as $M\to\infty$, by (3.1).

Lastly,

$$Z_2\leqslant A\cdot\sum\frac{|t_n^1|^k}{n}<A,$$

by (3.1).

This proves (3.4).

Thus the proof of the theorem is complete.

4. Recently, Mazhar [3] has proved the following theorem:

THEOREM M. *If $\{\lambda_n\}$ is a convex sequence such that $\sum\lambda_n/n<\infty$ and*

$$\sum_{\nu=1}^{n}\frac{|S_\nu|^k}{\nu}=O(\log n)\qquad(k\geqslant 1),\tag{4.1}$$

then $\sum a_n\lambda_n$ is summable $|C,1|_k$.

With the application of Theorem 1 we note the corresponding theorem for $|N,1/(n+1)|_k$ summability of a factored series ($k\geqslant 1$):

THEOREM 2. *If $\{\lambda_n\}$ is a convex sequence such that $\sum\lambda_n/n<\infty$ and condition* (4.1) *holds, then*

$$\sum\frac{a_n\lambda_n\log(n+1)}{n}$$

is summable $|N,1/(n+1)|_k$.

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