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ON DECOMPOSITION IN BARRELLED SPACES

BY

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1. Introduction and terminology. Throughout we assume, unless the contrary is specified, that X is a Hausdorff locally convex space (LCS) with the topology $\mathcal{F}$ and, when it is necessary to emphasize a particular locally convex topology $\mathcal{F}$ on X, we shall write X as $(X, \mathcal{F})$. We write X^* for the topological dual of $(X, \mathcal{F})$. If X and Y are vector spaces forming a dual pair, we use the symbols $\sigma(X, Y)$ and $\beta(X, Y)$ for the weak and the strong topologies on X, respectively, and denote by X^{**} the topological dual of $\big(X^*, \beta(X^*, X)\big)$. Let, further, $\varepsilon(X^{**}, X^*)$ denote the topology of uniform convergence on all $\mathcal{F}$-equicontinuous subsets of X^*, which is a locally convex topology on X^{**}, and let J denote the usual canonical embedding of X into X^{**}, i.e., $J(x)(f) = f(x)$ for all $x \in X$ and $f \in X^*$.

Let $\{M_i\}$ be a sequence of non-trivial subspaces of $(X, \mathcal{F})$. We say that $\{M_i\}$ is an *$\mathcal{F}$-basis* of subspaces ($\mathcal{F}$-bos) or an $\mathcal{F}$-decomposition for $(X, \mathcal{F})$ if to each $x \in X$ there corresponds a unique sequence $\{x_i\}$, $x_i \in M_i$, such that

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i,$$

the convergence of the infinite series being with respect to the topology $\mathcal{F}$. Given an $\mathcal{F}$-bos $\{M_i\}$ in $(X, \mathcal{F})$, there exists a sequence $\{P_i\}$ of orthogonal projections on X defined by

$$P_i(x) = x_i, \quad \text{where } x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \text{ and } x_i \in M_i,\ i \geqslant 1.$$

Then, for all $x \in X$, $P_i^2(x) = P_i(x)$ and $P_i\big(P_j(x)\big) = 0$ if $i \neq j$. If, for a given $\mathcal{F}$-bos $\{M_i\}$ for $(X, \mathcal{F})$, each projection P_i is continuous on $(X, \mathcal{F})$, then $\{M_i\}$ is called an *$\mathcal{F}$-Schauder basis* of subspaces ($\mathcal{F}$-Sbos) or an *$\mathcal{F}$-Schauder decomposition.*

Since the adjoints of projections are again projections on the dual space, one may be tempted to know if the dual X^* of an LCS X, having a Sbos, has also a Sbos. In this note* we are interested in the study of the existence of a Sbos in the topological dual X^* of an LCS X which has a Sbos and *vice versa*.

In the following section we come across problems pertaining to $\sigma(X^*, X)$ and $\beta(X^*, X)$-continuous linear operators and their adjoints as well as to certain characterizations of $\mathscr{F}$- and $\beta(X^*, X)$-Sbos in X and X^*, respectively.

2. Adjoints on X^*. We start with the following proposition, whose trivial proof is omitted:

PROPOSITION 2.1. *If $(X, \mathscr{F})$ is an LCS, then the linear map*

$$J:\ \big(X, \sigma(X, X^*)\big) \to \big(J(X), \sigma(J(X), X^*)\big)$$

is a topological isomorphism.

It is clear that $\varepsilon(X^{**}, X^*) \subset \beta(X^{**}, X^*)$, and if X is infrabarrelled, then $\varepsilon(X^{**}, X^*) = \beta(X^{**}, X^*)$ and conversely ([1], p. 229). Consequently, we have

PROPOSITION 2.2. *If X is infrabarrelled, then J is a topological isomorphism from $(X, \mathscr{F})$ into $\big(X^{**}, \beta(X^{**}, X^*)\big)$ or onto $\big(J(X), \beta(J(X), X^*)\big)$.*

Let X and Y be two vector spaces forming a dual pair with respect to the bilinear form $\langle x, y\rangle$, $x \in X$, $y \in Y$. Suppose that T: $X \to X$ is a linear operator. We may define another linear operator T^*: $Y \to Y$ by the relation

$$\langle x, T^* y\rangle = \langle Tx, y\rangle .$$

Let now X be equipped with a locally convex topology $\mathscr{F}$, let X^* be the topological dual of $(X, \mathscr{F})$ and let Y be the algebraic dual of X. Then $X^* \subset Y$. Suppose now $\mathscr{G}$ is any other locally convex topology on X compatible (see [1], p. 198) with the pairing of X and X^*. Then under T^*, as defined above, elements of X^* go to X^*, i.e., $T^*(X^*) \subset X^*$. Now let X^* be equipped with a locally convex topology τ such that τ is compatible with the pairing of X^* and X. Then it is clear that the restriction of T^* on X^* is continuous on (X^*, τ).

PROPOSITION 2.3. *Consider an LCS $(X, \mathscr{F})$ and a linear operator T: $X^* \to X^*$ such that T is continuous on $\big(X^*, \beta(X^*, X)\big)$. Then the adjoint map T^* of T is continuous on $\big(X^{**}, \beta(X^{**}, X^*)\big)$ and takes X^{**} into X^{**}.*

Proof. By the remark preceding Proposition 2.3, it is clear that the adjoint T^* exists on the algebraic dual of X^* and that $T^*(X^{**}) \subset X^{**}$.

To prove the continuity of T^*, let $F_\alpha \to 0$ in $\beta(X^{**}, X^*)$. Let A be an arbitrarily chosen $\sigma(X^*, X^{**})$-bounded set in X^*. Since $\beta(X^*, X)$

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and $\sigma(X^*, X^{**})$ are compatible topologies on X^*, it follows from Mackey's Theorem ([1], p. 209) that A is $\beta(X^*, X)$-bounded, and since T is $\beta(X^*, X)$-continuous on X^*, $T(A)$ is $\beta(X^*, X)$-bounded on X^* which implies, by Mackey's Theorem once again, that $T(A)$ is $\sigma(X^*, X^{**})$-bounded on X^*. Thus

$$F_a \in [T(A)]^0 \quad \text{for all } a \geqslant a_0 = a_0(A),$$

whence

$$|F_a(T(f))| \leqslant 1 \quad \text{for all } f \in A \text{ and } a \geqslant a_0,$$

$$|T^*(F_a)(f)| \leqslant 1 \quad \text{for all } f \in A \text{ and } a \geqslant a_0,$$

$$T^*(F_a) \in A^0 \quad \text{for all } a \geqslant a_0,$$

where A^0 is the polar (see [1], p. 190) of A in X^{**}; since A is arbitrary, it follows that $T^*(F_a) \to 0$ in $(X^{**}, \beta(X^{**}, X^*))$ and the proof of the proposition is completed.

PROPOSITION 2.4. *Let $(X, \mathscr{F})$ be an LCS, and X^* the topological dual of $(X, \mathscr{F})$. If T is a $\sigma(X^*, X)$-continuous linear operator on X^* and T^* is the adjoint of T defined on the algebraic dual of X^*, then $T^*(J(X)) \subset J(X)$.*

Proof. Let $F \in T^*(J(X))$ be such that there is an $x \in X$ with $F = T^*(J(x))$. As T is $\sigma(X^*, X)$-continuous, there exists a $y \in X$, depending on x, such that

$$|T(f)(x)| \leqslant |f(y)| \quad \text{for all } f \in X^*.$$

Therefore

$$|J(x)(T(f))| \leqslant |f(y)| \quad \text{for all } f \in X^*,$$

whence

$$|T^*(J(x))(f)| \leqslant |f(y)| \quad \text{for all } f \in X^*$$

and

$$|F(f)| \leqslant |f(y)| \quad \text{for all } f \in X^*.$$

Thus F is a $\sigma(X^*, X)$-continuous linear functional on X^*, and so $F \in J(X)$, which completes the proof.

PROPOSITION 2.5. *Let $(X, \mathscr{F})$ be an LCS. Then every $\sigma(X^*, X)$-continuous linear operator T on X^* into itself is $\beta(X^*, X)$-continuous.*

Proof. Let U be a $\beta(X^*, X)$-neighbourhood of $0 \in X^*$. Then there exists a bounded set A in X such that $A^0 \subset U$. If T^* is the adjoint operator of T, if follows that T^* is $\sigma(X^{**}, X^*)$-continuous. By Proposition 2.1, $J(A)$ is $\sigma(X^{**}, X^*)$-bounded, and so is $T^*J(A)$. Since T is $\sigma(X^*, X)$-continuous, by Proposition 2.4 we have $T^*J(X) \subset J(X)$, and so $T^*J(A) \subset J(X)$. Set $B = J^{-1}T^*J(A)$. Then it follows from Proposition 2.1 that B is $\sigma(X, X^*)$-bounded, which implies that B^0 is a $\beta(X^*, X)$-neighbourhood of $0 \in X^*$. Now it is sufficient to show that $B^0 \subset T^{-1}(U)$. Take $f \in B^0$. Then

$$|f[J^{-1}T^*J(x)]| \leqslant 1 \quad \text{for all } x \in A,$$

$$|T^*J(x)(f)| \leqslant 1 \text{ for all } x \in A, \quad |J(x)(Tf)| \leqslant 1 \text{ for all } x \in A,$$

$$|Tf(x)| \leqslant 1 \text{ for all } x \in A, \quad Tf \in A^0 \subset U, \quad T(B^0) \subset U.$$

PROPOSITION 2.6. *If $(X, \mathscr{F})$ is an infrabarrelled space and T is a linear operator on X^* into itself such that T is $\sigma(X^*, X)$-continuous, then $E = J^{-1}T^*J$ is an $\mathscr{F}$-continuous linear operator on X into itself.*

Proof. Propositions 2.3 and 2.5 ensure the existence and continuity of T^* on $(X^{**}, \beta(X^{**}, X^*))$. From Proposition 2.4 it is clear that the range of E is in X. Also, X is infrabarrelled, and so J is a topological isomorphism in the sense of Proposition 2.2. The result now easily follows.

Remark. Following the proof of Proposition 2.6 we can easily see that if T is a projection on X^*, then E (defined as above) is also a projection on X.

We also need the following result:

PROPOSITION 2.7. *Let $\langle X, Y\rangle$ be a dual pair of vector spaces. If $\varkappa$ is a collection of $\sigma(X, Y)$-continuous linear maps from X into itself such that, for each $x \in X$, the set $\{f(x): f \in \varkappa\}$ is $\beta(X, Y)$-bounded, then $\varkappa$ is a $\beta(X, Y)$-equicontinuous family.*

Proof. The proof follows by an easy application of the well-known bipolar theorem ([1], p. 92). Indeed, let A be a $\sigma(Y, X)$-bounded set. Then A^0 is a $\beta(X, Y)$-neighbourhood of $0 \in X$. Clearly, $B = \bigcap \{f^{-1}[A^0]: f \in \varkappa\}$ is balanced, convex, $\sigma(X, Y)$-closed and absorbing. Hence $B = B^{00}$ and the result is proved.

The following proposition has been proved in [2].

PROPOSITION 2.8. *Let X be a complete Hausdorff topological vector space whose topology $\mathscr{F}$ is generated by a family $\{p_\lambda: \lambda \in D\}$ of pseudo-norms, D being a directed set. Suppose further $\{M_n\}$ is a sequence of non-trivial closed subspaces of X such that*

$$\Big[\bigcup_{n=1}^{\infty} M_n\Big] = X.$$

If, for every $\lambda \in D$, there exists a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda\Big(\sum_{i=1}^{m} x_i\Big) \leqslant K_\lambda p_\mu\Big(\sum_{i=1}^{n} x_i\Big)$$

for all integers $m, n \geqslant 1$, $m \leqslant n$, and for all sequences $\{x_i\}$ of X with $x_i \in M_i$, then $\{M_n\}$ is a Schauder decomposition for X.

Note. If each pseudo-norm in Proposition 2.8 is replaced by a paranorm, the above result is due to Russo [5]; however, his proof is still different from ours.

Proposition 2.8 may be used to derive

PROPOSITION 2.9. *If $\{E_i\}$ is a sequence of continuous, non-trivial, orthogonal projections of a complete barrelled space X into itself and if $\{\sum_{i=1}^{n} E_i(x)\}$ is bounded for each $x \in X$, then $\{R(E_i)\}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R(E_i)]$ and $\{R(E_i^*)\}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R(E_i^*)]$ in $\beta(X^*, X)$.*

Proof. Let

$$Q_n(x) = \sum_{i=1}^{n} E_i(x).$$

From hypothesis, $\{Q_n\}$ is a non-trivial sequence of continuous projections which is also pointwise bounded on X and so, by the barrel theorem ([3], p. 104), is equicontinuous on X. Let now D denote the family of all continuous seminorms on X. Then, for a given $p \in D$, there exists a $K_p > 0$ and a $q \in D$ such that

$$p(Q_n(x)) \leqslant K_p q(x) \quad \text{for all } n \geqslant 1 \text{ and } x \in X. \tag{1}$$

Choose integers m and n with $1 \leqslant m \leqslant n$ and let

$$x = \sum_{i=1}^{n} x_i, \qquad x_i \in R(E_i) \ (1 \leqslant i \leqslant n).$$

Then, by (1),

$$p\Big(\sum_{i=1}^{m} x_i\Big) = p\Big(Q_m\Big(\sum_{i=1}^{n} x_i\Big)\Big) \leqslant K_p q\Big(\sum_{i=1}^{n} x_i\Big),$$

and we get the first part of the proposition from Proposition 2.8.

To prove the second part, let

$$Q_n^* = \sum_{i=1}^{n} E_i^*,$$

where E_i^* is adjoint of E_i. It follows that Q_n^* is $\sigma(X^*, X)$-continuous. Observe that $\{Q_n(x)\}$ is bounded for each $x \in X$, and so $\{f(Q_n(x))\}$ is bounded for each $x \in X$ and each $f \in X^*$. Thus $\{Q_n^*(f)\}$ is $\sigma(X^*, X)$-bounded for each $f \in X^*$. Since X is barrelled, $\{Q_n^*(f)\}$ is $\beta(X^*, X)$-bounded. Thus, by Proposition 2.7, $\{Q_n^*\}$ is $\beta(X^*, X)$-equicontinuous. The rest of the proof now follows as the first part did.

3. The main theorem. We are now ready to state the main result of this paper:

THEOREM 3.1. (i) *Let* $(X, \mathscr{F})$ *be an LCS. If* $\{M_i, E_i\}$ *is an* $\mathscr{F}$*-Sbos for* X, *then* $\{R(E_i^*); E_i^*\}$ *is a* $\sigma(X^*, X)$*-Sbos for* X^*.

(ii) *Let* $(X, \mathscr{F})$ *be a complete barrelled space. If* $\{N_i, P_i\}$ *is a* $\sigma(X^*, X)$*-Sbos for* X^*, *then* $\{R(E_i), E_i\}$, *where* $E_i = J^{-1}P_i^* J$, *is an* $\mathscr{F}$*-Sbos for* X.

Proof. (i) The proof is straightforward and is therefore omitted.

(ii) By Proposition 2.6, each E_i is a continuous projection on $(X, \mathscr{F})$. Now from hypothesis, for each $f \in X^*$,

$$f = \lim_{n\to\infty} \sum_{i=1}^{n} P_i(f) \qquad (\text{in } \sigma(X^*, X)),$$

whence

$$f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} P_i(f)(x) \qquad \text{for each } x \in X \text{ and } f \in X^*.$$

By Proposition 2.4, for each $x \in X$ there exists $y_i \in X$ with $P_i^*(J(x)) = J(y_i)$. Thus

$$E_i(x) = J^{-1}P_i^* J(x) = y_i.$$

Also

$$P_i(f)(x) = J(x)(P_i(f)) = P_i^*(J(x))(f) = J(y_i)(f) = f(y_i) = f(E_i(x)).$$

Hence, for each $x \in X$ and each $f \in X^*$,

$$f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(E_i(x)),$$

and so the sequence $\{\sum_{i=1}^{n} E_i(x)\}$ is $\sigma(X, X^*)$-bounded. Therefore, by the quoted Mackey's theorem, it is also $\mathscr{F}$-bounded, and hence, by Proposition 2.9, the required result follows, provided we could show that

$$X_0 \equiv [\bigcup_{i=1}^{\infty} R(E_i)] = X.$$

However, this easily follows from the Hahn-Banach Theorem. Indeed, let $x \in X$ and $x \notin X_0$. By the Hahn-Banach Theorem there exists an $f \in X^*$ with $f(x) = 1$, and $f(y) = 0$ for all $y \in X_0$. Since $E_i(x) \in X_0$ for each $i \geqslant 1$, we have

$$1 = f(x) = \sum_{i=1}^{\infty} f(E_i(x)) = 0,$$

but this is absurd and so $x = X_0$. The proof is completed.

Theorem 3.1 may be used to derive the following result:

THEOREM 3.2. *Let* $(X, \mathcal{F})$ *be a complete barrelled space. If* $\{N_i, P_i\}$ *is a* $\sigma(X^*, X)$*-Sbos for* X^**, then* $\{N_i, P_i\}$ *is a* $\beta(X^*, X)$*-Sbos for* $[\bigcup_{i=1}^{\infty} R(P_i)]$.

Proof. Let $E_i = J^{-1}P_i^* J$. Then, by Theorem 3.1 (ii), $\{R(E_i), E_i\}$ is an $\mathcal{F}$-Sbos for X and so, by Proposition 2.9, $\{R(E_i^*)\}$ is a $\beta(X^*, X)$-Sbos for $[\bigcup_{i=1}^{\infty} R(E_i^*)]$. We need to show now that $E_i^* = P_i$ for each $i \geqslant 1$. To prove it, let $x \in X$ and $f \in X^*$ be chosen arbitrarily but fixed. By Proposition 2.4, $P_i^*(J(X)) \subset J(X)$. Then there exists a $y_i \in X$ with $P_i^* J(x) = J(y_i)$. Hence

$$\begin{aligned} E_i^*(f)(x) &= f(E_i(x)) = f(J^{-1}P_i^* J(x)) = f(y_i) \\ &= J(y_i)(f) = P_i^*(J(x))(f) = J(x)(P_i(f)) = P_i(f)(x). \end{aligned}$$

Thus $E_i^*(f) = P_i(f)$ and the proof is completed.

Remark. Theorems 3.1 and 3.2 generalize similar results in normed spaces proved by Retherford [4].

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