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ON INVERSES OF ALMOST CONTINUOUS BIJECTIONS

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The notions of nearly continuity/openness and almost continuity/openness stem from functional analysis; they appear in context of the open mapping, the closed graph and the Souslin-graph theorems, and also of the purely topological Blumberg's theorem (see e.g. [2], [4], [5], and references therefrom).

A mapping $f\colon X \to Y$, where X and Y are topological spaces, is said to be *nearly continuous* (resp. *nearly open*) at x if for every open neighbourhood V of $f(x)$ (resp. U of x)

$$x \in \operatorname{Int} \overline{f^{-1}(V)} \qquad (\text{resp.}\, f(x) \in \operatorname{Int} \overline{f(U)});$$

f is *almost continuous* (resp. *almost open*) at x if for every such V (resp. U)

$$x \in \operatorname{Int} D(f^{-1}(V)) \qquad (\text{resp.}\, f(x) \in \operatorname{Int} D(f(U))).$$

($D(A)$ denotes – as in [1] – the set of all points of the underlying space at which A is of second category.) The resulting sets of points of nearly continuity (nearly openness) and almost continuity (almost openness) of f are denoted – respectively – by $C_n(f)$ ($O_n(f)$) and $C_a(f)$ ($O_a(f)$). Obviously

$$C_n(f) \supset C_a(f) \quad \text{and} \quad O_n(f) \supset O_a(f).$$

If Y has a countable base, then $C_a(f)$ is residual in X, i.e. its complement is of first category (cf. [4], Theorem 1). On the other hand, almost continuity is a rather strong property: each almost continuous mapping $f\colon X \to Y$ having the Baire property is continuous provided that Y is regular ([4], Theorem 4).

In Theorem 3 of [6] we constructed a nearly continuous bijection $f\colon R \to R$ such that the set $O_n(f) \cup C_n(f^{-1})$ was co-dense. Here we give an example of a similar kind – an almost continuous (hence nearly continuous) bijection $f\colon R \to R$ such that the set $C_n(f^{-1})$ is of measure zero. Both examples contrast with the fact that each continuous real bijection is a homeomorphism.

The following lemma is a convenient version of the Sierpiński–Lusin theorem [3].

LEMMA 1. *Each second category set $A \subset R$ can be represented as a disjoint sum $\bigcup_1^\infty A_k$, where $D(A) = D(A_k)$ for all $k \in N$.*

Proof. Define $G = \operatorname{Int} D(A)$; since $A \setminus G$ is of first category (cf. [1], 1.10.VI),

$$D(A) = D(A \cap G) = \bar{G}.$$

Let $G = \bigcup_{i \in N_0} G_i$, where $\emptyset \neq N_0 \subset N$ and G_i are pairwise disjoint non-empty open segments. We have

$$D(A \cap G_i) = \overline{G_i};$$

by the Sierpiński–Lusin theorem (cf. [3], p. 115 and 176), each set $A \cap G_i$ is of the form $\bigcup_{k=1}^{\infty} A_{ik}$, where

$$A_{ik} \cap A_{ik'} = \emptyset \quad \text{whenever } k \neq k'$$

and

$$D(A_{ik}) = \overline{G_i} \quad \text{for all } k \in N.$$

Put

$$A_1 = \bigcup_{i \in N_0} A_{i1} \cup (A \setminus G)$$

and

$$A_k = \bigcup_{i \in N_0} A_{ik} \quad \text{for } k \geqslant 2;$$

these sets satisfy the requirements.

When we say that C is a Cantor set we mean that C is a subset of R which is a homeomorphic copy of the Cantor ternary set of the closed unit interval I.

LEMMA 2. *Each Cantor set of positive measure contains a Cantor set of measure zero.*

Proof is an easy exercise.

Given a measurable set A in R, we define

$$P(A) = \{x \in R\colon\ m(A \cap U) > 0 \text{ for every open } U \ni x\},$$

where m stands for the Lebesgue measure; it is a measure-theoretic analogon of the set $D(A)$.

LEMMA 3. *Suppose $X \subset R$ is a second category set of power continuum and $Y \subset R$ is a non-empty Borel set such that*

$$Y \subset D(Y) \quad \textit{or} \quad Y \subset P(Y).$$

Then there exists a bijection $f\colon X \to Y$ such that for every open set V in R

$$V \cap Y \neq \emptyset \quad \textit{implies} \quad D(f^{-1}(V \cap Y)) = D(X).$$

Proof. By Lemma 1, X is of the form $\bigcup_{k=0}^{\infty} A_k$, where $D(X) = D(A_k)$ for all k; we may additionally assume that A_0 is of power continuum.

Let $\{V_k\colon k \in N\}$ be an open base for the subspace Y of R, consisting of non-empty sets. Since V_1 is an uncountable Borel set in R (of second category or of positive measure), it contains a Cantor set C_1 of measure zero, by the Alexandroff–Hausdorff theorem (cf. [1], 3.37.I) and Lemma 2. For the same reason, $V_2 \setminus C_1$ contains a Cantor set C_2 of measure zero. Continuing this process we get a pairwise disjoint sequence $\{C_k\}$ of Cantor sets of measure zero in Y with

$$C_k \subset V_k \setminus \bigcup_{i=1}^{k-1} C_i \quad \text{for } k \in N.$$

Let f_k $(k \in N)$ be an arbitrary injection of A_k into C_k. Put

$$C_0 = Y \setminus \bigcup_{1}^{\infty} f_k(A_k);$$

C_0 is of power continuum, because $Y \setminus \bigcup_{1}^{\infty} C_k$ is a Borel set which is of second category or of positive measure. There is a bijection $f_0\colon A_0 \to C_0$. Define $f(x) = f_k(x)$ whenever $x \in A_k$ $(k \geqslant 0)$. For any index $k \geqslant 1$ we have

$$D(f^{-1}(V_k)) \supset D(f^{-1}(C_k)) \supset D(A_k) = D(X),$$

which yields the assertion.

THEOREM. *There exists an almost continuous bijection f of R onto itself such that the set $C_n(f^{-1})$ is of measure zero.*

Proof. Put $X_k = [2k-1, 2k]$ for $k \in N$, $X_0 = R \setminus \bigcup_{1}^{\infty} X_k$. Choose a disjoint sequence $\{Y_k\colon k \in N\}$ of Cantor sets in R with $Y_k \subset P(Y_k)$ and $m(Y_0) = 0$, where $Y_0 = R \setminus \bigcup_{1}^{\infty} Y_k$. By Lemma 3, there exists a bijection f_0 of X_0 onto Y_0 (which is residual) such that for every open set V in R

$$V \cap Y_0 \neq \emptyset \quad \text{implies} \quad D(f_0^{-1}(V \cap Y_0)) = D(X_0),$$

and for each $k \in N$ there exists a bijection $f_k\colon X_k \to Y_k$ such that for every open V

$$V \cap Y_k \neq \emptyset \quad \text{implies} \quad D(f_k^{-1}(V \cap Y_k)) = D(X_k).$$

Let f denote the bijection of R onto itself built up of all those partial bijections. If $x \in X_0$ and V is an open neighbourhood of $f(x)$, then

$$D(f^{-1}(V)) \supset D(f_0^{-1}(V \cap Y_0)) \supset X_0,$$

which shows that $x \in C_a(f)$ (X_0 is open). If $x \in X_k$ ($k \geqslant 1$) and $f(x) \in V$ (open), then $V \cap Y_0 \neq \emptyset$ and

$$D(f^{-1}(V)) \supset D(f_k^{-1}(V \cap Y_k) \cup f_0^{-1}(V \cap Y_0)) \supset X_k \cup X_0,$$

which again yields $x \in C_a(f)$ (because $x \in \operatorname{Int}(X_k \cup X_0)$). Thus $C_a(f) = R$. Since each set $f(\operatorname{Int} X_k)$ is nowhere dense,

$$O_n(f) \subset R \setminus \bigcup_1^{\infty} \operatorname{Int} X_k.$$

Hence

$$C_n(f^{-1}) = f(O_n(f)) \subset Y_0 \cup f(N),$$

where the set on the right is of measure zero.

It is possible to modify the proof so that to get a bijection $f\colon I \to I$ with the same properties. To this aim, put $J = (0, 1]$, $I_k = [1/(2k), 1/(2k-1)]$ and $X_0 = J \setminus \bigcup_1^{\infty} I_k$. Choose X_k ($k \in N$) so that:

X_k contains a Cantor set (which can be previously chosen),
$I_k = X_{2k-1} \cup X_{2k} = D(X_{2k-1}) = D(X_{2k})$,
$X_{2k-1} \cap X_{2k} = \emptyset$ for all k.

Choose Cantor sets $Y_k \subset J$ so that $Y_k \subset P(Y_k)$, $m(Y_0) = 0$ (where $Y_0 = J \setminus \bigcup_1^{\infty} Y_k$) and $\limsup_k Y_{2k} = 0$. Now we find $f_k\colon X_k \to Y_k$ ($k \geqslant 0$) with the help of Lemma 3 and get a bijection $f\colon J \to J$ such that $C_a(f) = J$ and $C_n(f^{-1})$ is of measure zero. We extend f on I defining $f(0) = 0$; given a neighbourhood V of 0, there is an index k_0 such that

$$Y_{2k} \subset V \quad \text{for } k \geqslant k_0,$$

and so

$$\begin{aligned} D(f^{-1}(V)) &\supset D(f_0^{-1}(V \cap Y_0) \cup f_{2k}^{-1}(Y_{2k})) \\ &= D(X_0) \cup D(X_{2k}) \supset X_0 \cup I_k \quad \text{for } k \geqslant k_0, \end{aligned}$$

which shows that $0 \in C_a(f)$.

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