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ON ADMISSIBLE WHITNEY MAPS

BY

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1. Introduction. Throughout this paper, the word *compactum* means a compact metric space. A *continuum* is a connected compactum. Let X be a metric space with metric ϱ. The *hyperspaces* of X are the spaces

$$2^X = \{A \mid A \text{ is a nonempty and compact subset of } X\}$$

and

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\}$$

which are metrized with the *Hausdorff metric* ϱ_H, i.e.,

$$\varrho_H(A, B) = \max\{\sup_{a\in A} \varrho(a, B), \sup_{b\in B} \varrho(b, A)\}.$$

A *Whitney map* for a hyperspace $\mathscr{H} = 2^X$ or $C(X)$ is a continuous function $\omega\colon \mathscr{H} \to [0, \omega(X)]$ such that $\omega(\{x\}) = 0$ for each $x \in X$, and if $A, B \in \mathscr{H}$, $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$. In [12], Whitney showed that for any metric space (X, ϱ) there always exists a Whitney map for $\mathscr{H} = 2^X$ or $C(X)$. We recall the construction. Let $A \in \mathscr{H}$. For each $n \geq 2$ let

$$F_n(A) = \{K \subset A \mid K \neq \emptyset \text{ and the cardinality of } K \text{ is } \leq n\}.$$

Also, define $\lambda_n\colon F_n(A) \to [0, \infty)$ by letting

$$\lambda_n(\{a_1, a_2, \ldots, a_n\}) = \min\{\varrho(a_i, a_j) \mid i \neq j\}$$

for all $\{a_1, a_2, \ldots, a_n\} \in F_n(A)$, and let

$$\omega_n(A) = \sup \lambda_n(F_n(A)).$$

Then

$$(*) \qquad \omega(A) = \sum_{n=2}^{\infty} \omega_n(A)/2^{n-1}.$$

The notion of Whitney map is a convenient and important tool in order to study hyperspaces theory. It is of interest to obtain information about Whitney levels $\omega^{-1}(t)$ $(0 < t < \omega(X))$ and to determine those properties which are preserved by the convergence of positive Whitney levels $\omega^{-1}(t_n)$ $(t_n > 0)$ to the zero level $\omega^{-1}(0) = X$. In [3] and [11], Curtis, Schori and West proved that for any Peano continuum (= locally connected continuum) X, 2^X is homeomorphic to the Hilbert cube $Q = [-1, +1]^\infty$, and if X contains no free arc, $C(X)$ is homeomorphic to Q. In [5], Goodykoontz and Nadler introduced the notion "admissible Whitney map". A Whitney map ω for $\mathscr{H} = 2^X$ or $C(X)$ is *admissible* [5] if there is a homotopy $h\colon \mathscr{H} \times I \to \mathscr{H}$ satisfying the following conditions:

(A1) for all $A \in \mathscr{H}$,

$$h(A, 1) = A, \quad h(A, 0) \in F_1(X) = \{\{x\} \mid x \in X\};$$

(A2) if $\omega(h(A, t)) > 0$ for some $A \in \mathscr{H}$, $t \in I$, then

$$\omega(h(A, s)) < \omega(h(A, t)) \quad \text{whenever } 0 \leqslant s < t \leqslant 1.$$

In [5], it was shown that if X is either a compact starshaped subset of a Banach space or a 1-dimensional AR (= dendrite), then there exist admissible Whitney maps for $\mathscr{H} = 2^X$ and $C(X)$. By using the notion of admissible Whitney map, Goodykoontz and Nadler [5] proved the following

(1.1) *Let X be a Peano continuum and let ω be an admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$. If $\mathscr{H} = C(X)$, assume that X contains no free arc. Then, for any $t \in (0, \omega(X))$, $\omega^{-1}(t)$ is a Hilbert cube.*

A Whitney map ω for $\mathscr{H} = 2^X$ or $C(X)$ is *strongly admissible* [6] if there is a homotopy $h\colon \mathscr{H} \times I \to \mathscr{H}$ satisfying (A1), (A2) and

(A3) $h(\{x\}, t) = \{x\}$ for each $x \in X$ and $t \in I$.

In [6], we proved the following

(1.2) *Let X be a Peano continuum and let ω be an admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$. If $\mathscr{H} = C(X)$, assume that X contains no free arc. Then the restriction*

$$\omega | \omega^{-1}((0, \omega(X)))\colon \omega^{-1}((0, \omega(X))) \to (0, \omega(X))$$

of ω to $\omega^{-1}((0, \omega(X)))$ is a trivial bundle map with Hilbert cube fibers. Moreover, if X is the Hilbert cube Q, then there is a Whitney map ω for $\mathscr{H} = 2^Q$ or $C(Q)$ such that $\omega|\omega^{-1}([0, \omega(Q)))$ is a trivial bundle map with Hilbert cube fibers. Also, if X is the n-sphere S^n $(n \geqslant 1)$, then there is a Whitney map ω for $\mathscr{H} = 2^{S^n}$ $(n \geqslant 1)$ or $C(S^n)$ $(n \geqslant 2)$ such that, for some $t_0 \in (0, \omega(S^n))$, $\omega|\omega^{-1}((0, t_0))$ is a trivial bundle map with $S^n \times Q$ fibers.

The purpose of this paper is to prove the following:

(1) Let P be a finite collapsible polyhedron and let $\mathscr{H} = 2^P$ or $C(P)$. If $\mathscr{H} = C(P)$, assume that P contains no free arc. Then there is a Whitney map ω for $\mathscr{H}$ such that $\omega|\omega^{-1}((0, \omega(P)))$ is a trivial bundle map with Hilbert cube fibers.

(2) Let K be a cubical complex and let $P = |K|$. Let $\mathscr{H} = 2^P$ or $C(P)$. If $\mathscr{H} = C(P)$, assume that P contains no free arc. If K is locally regular collapsible, then there is a Whitney map ω for $\mathscr{H}$ such that, for some $t_0 \in (0, \omega(P))$, $\omega|\omega^{-1}((0, t_0))$ is a trivial bundle map with $P \times Q$ fibers.

2. Whitney maps and hyperconvex metric spaces. A metric space (X, ϱ) is *hyperconvex* (or *injective*) [1] if ϱ is convex and any collection of solid spheres in pairwise intersection in X has a common point. In [1], Theorem 3, it was proved that a metric space (X, ϱ) is hyperconvex if and only if every mapping which increases no distance from a subset of any metric space Y to X can be extended, increasing no distance, over Y.

First, we show the following

(2.1) THEOREM. *Let (X, ϱ) be a hyperconvex metric compactum. Suppose that ω is the Whitney map for $\mathscr{H} = 2^X$ or $C(X)$ which is defined by* (∗) *and the metric ϱ. Then ω is strongly admissible.*

Proof. It is well known that there exists a Banach space B with norm $\|\cdot\|$ such that B contains X and, for any $x, y \in X$, $\varrho(x, y) = \|x-y\|$ (see [8]). Let ω' be the Whitney map for $\mathscr{H}' = 2^B$ or $C(B)$ as is defined by (∗) and the metric ϱ', where $\varrho'(b, b') = \|b-b'\|$ for $b, b' \in B$. Since X is an AR (see [1]), there is a retraction $r\colon \mathscr{H} \to X$, i.e., $r(\{x\}) = x$ for each $x \in X$. Let $A \in \mathscr{H}$. Define a homotopy $h_A\colon A \times I \to B$ by

$$h_A(a, t) = (1-t)\cdot r(A) + t\cdot a \quad \text{for each } a \in A,\ t \in I. \tag{1}$$

Also, define a homotopy $h'\colon \mathscr{H} \times I \to \mathscr{H}'$ by

$$h'(A, t) = \{h_A(a, t) \mid a \in A\} \quad \text{for each } A \in \mathscr{H},\ t \in I. \tag{2}$$

Since (X, ϱ) is hyperconvex, there is a contraction $f\colon B \to X$, i.e.,

$$\varrho(f(y), f(z)) \leqslant \varrho'(y, z) = \|y-z\| \quad \text{for } y, z \in B.$$

If $x, y \in A$, $x \neq y$, and $0 \leqslant t' < t \leqslant 1$, then

$$\begin{aligned} \varrho'(h_A(x, t'), h_A(y, t')) &= \|t'(x-y)\| < \|t(x-y)\| \\ &= \varrho'(h_A(x, t), h_A(y, t)). \end{aligned} \tag{3}$$

Hence, if A is nondegenerate, by (3) we have

$$\omega'(h'(A, t')) < \omega'(h'(A, t)) \quad \text{for } 0 \leqslant t' < t \leqslant 1 \tag{4}$$

(see [5], (2.13)). Since f is a contraction, we can easily see

$$\omega(f(h'(A, t))) = \omega'(f(h'(A, t))) \leqslant \omega'(h'(A, t)) \quad \text{for each } A \in \mathscr{H},\ t \in I. \tag{5}$$

Consider the function $K_\varrho\colon \mathscr{H} \times [0, \infty) \to \mathscr{H}$ defined by

$$K_\varrho(A, s) = \{x \in X \mid \varrho(A, x) \leqslant s\} \quad \text{for each } A \in \mathscr{H},\ s \in [0, \infty).$$

Since ϱ is convex, K_ϱ is continuous (see [5], (1.2)). For each $A \in \mathscr{H}$ and $t \in I$, there is the minimal number $m(A, t) \geqslant 0$ such that

$$\omega\big(K_\varrho\big(f(h'(A, t)), m(A, t)\big)\big) = \omega'(h'(A, t)). \tag{6}$$

Define a homotopy $h\colon \mathscr{H} \times I \to \mathscr{H}$ by

$$h(A, t) = K_\varrho\big(f(h'(A, t)), m(A, t)\big) \quad \text{for each } A \in \mathscr{H},\ t \in I. \tag{7}$$

Clearly, we have $\omega(h(A, t)) = \omega'(h'(A, t))$. Hence (4) implies that ω satisfies the condition (A2). Obviously, ω satisfies the conditions (A1) and (A3). Thus ω is strongly admissible.

A metric space (X, ϱ) is *locally hyperconvex* if for any $x \in X$ there is a neighborhood U of x in X such that $\varrho | U$ is a hyperconvex metric. Then we have the following

(2.2) THEOREM. *Let (X, ϱ) be a locally hyperconvex metric continuum and let ω be the Whitney map for $\mathscr{H} = 2^X$ or $C(X)$ which is defined by $(*)$ and the metric ϱ. Then there exist a positive number $t_0 \in (0, \omega(X))$ and a homotopy*

$$h\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

such that

(A1)′ *$h(A, 1) = A$, $h(A, 0) \in F_1(X)$ for each $A \in \omega^{-1}([0, t_0))$;*

(A2)′ *if $\omega(h(A, t)) > 0$ for some $A \in \omega^{-1}((0, t_0))$ and $t \in I$, then*

$$\omega(h(A, s)) < \omega(h(A, t)) \quad \textit{whenever } 0 \leqslant s < t \leqslant 1;$$

(A3)′ *$h(\{x\}, t) = \{x\}$ for each $x \in X$ and $t \in I$.*

Proof. Since ϱ is locally hyperconvex, there is a positive number $\varepsilon > 0$ such that, for any $x \in X$, $\varrho | S(x, \varepsilon)$ is hyperconvex, where

$$S(x, \varepsilon) = \{y \in X \mid \varrho(x, y) \leqslant \varepsilon\}.$$

Also, there are points $x_1, x_2, \ldots, x_n$ of X such that

$$X = \bigcup_{i=1}^{n} \operatorname{Int} S(x_i, \varepsilon/3).$$

Since X is an ANR, there is a retraction $r\colon \mathscr{U} \to F_1(X) = X$, where $\mathscr{U}$ is a neighborhood of $F_1(X)$ in $\mathscr{H}$. Let δ be a positive number such that

$$(2^n + 1)\delta < \varepsilon/3.$$

Choose a sufficiently small positive number $t_0 \in (0, \omega(X))$ such that

(1) if $A \in \omega^{-1}([0, t_0))$, then $A \subset S(x_i, \varepsilon/3)$ for some i;

(2) $\omega^{-1}([0, t_0)) \subset \mathscr{U}$;

(3) if $A \in \omega^{-1}([0, t_0))$, then $S(r(A), \delta) \supset A$.

Set

$$\mathscr{A}_i = \{A \in \omega^{-1}([0, t_0))| \ A \subset S(x_i, \varepsilon/3)\},$$
$$\mathscr{B}_i = \{A \in \omega^{-1}([0, t_0))| \ A \subset \operatorname{Int} S(x_i, 2\varepsilon/3)\} \quad (i = 1, 2, \ldots, n),$$
$$\mathscr{A}_0 = \mathscr{B}_0 = \emptyset.$$

Let φ_i: $\omega^{-1}([0, t_0)) \to I$ $(i = 1, 2, \ldots, n)$ be a map such that

(4) $\varphi_i(A) = 0$ if $A \in \mathscr{A}_i$ and $\varphi_i(A) = 1$ if A is not contained in $\mathscr{B}_i$.

By induction, we shall construct a homotopy

$$h_i: \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0)) \quad (i = 0, 1, \ldots, n)$$

such that

A (i) $h_i(A, 1) = A$ for each $A \in \omega^{-1}([0, t_0))$;

B (i) if $A \in \bigcup_{j=0}^{i} \mathscr{A}_j$, then $h_i(A, 0) = r(A) \in F_1(X)$;

C (i) if $A \in \omega^{-1}([0, t_0))$ and $0 \leqslant s \leqslant t \leqslant 1$, then

$$\omega(h_i(A, s)) \leqslant \omega(h_i(A, t));$$

D (i) $h_i(\{x\}, t) = \{x\}$ for $x \in X$ and $t \in I$;

E (i) if $\omega(h_i(A, s)) = \omega(h_i(A, t))$ for some $A \in \omega^{-1}([0, t_0))$ and $s, t \in I$, then $h_i(A, s) = h_i(A, t)$;

F (i) $h_i(A, t) \subset S(r(A), 2^i\delta)$ for each $A \in \omega^{-1}([0, t_0))$ and $t \in I$.

First, for the case $i = 0$, define

$$h_0: \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

by $h_0(A, t) = A$ for each $A \in \omega^{-1}([0, t_0))$ and $t \in I$. Next, we suppose that there is a homotopy

$$h_i: \varphi^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

satisfying the conditions A (i)–F (i). We shall construct a homotopy

$$h_{i+1}: \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

satisfying the conditions A $(i+1)$–F $(i+1)$. If $A \in \mathscr{B}_{i+1}$, then (3) implies that $r(A) \in S(x_{i+1}, (2\varepsilon/3) + \delta)$. By F (i), we can see that if $A \in \mathscr{B}_{i+1}$, then

$$h_i(A, 0) \subset S(x_{i+1}, (2\varepsilon/3) + \delta(1 + 2^i)) \subset S(x_{i+1}, \varepsilon).$$

Since $\varrho|S(x_{i+1}, \varepsilon)$ is hyperconvex, by the proof of (2.1) we have a homotopy

$$h'_{i+1}: h_i(\mathscr{B}_{i+1} \times \{0\}) \times I \to \omega^{-1}([0, t_0))$$

such that

(5) $h'_{i+1}(A, 1) = A$, $h'_{i+1}(A, 0) = r(A)$ for each $A \in h_i(\mathscr{B}_{i+1} \times \{0\})$;

(6) if $\omega(h'_{i+1}(A, t)) > 0$ for some $A \in h_i(\mathscr{B}_{i+1} \times \{0\})$ and $t \in I$, then

$$0 < \omega(h'_{i+1}(A, s)) < \omega(h'_{i+1}(A, t)) \quad \text{whenever } 0 < s < t \leqslant 1;$$

(7) $h'_{i+1}(\{x\}, t) = \{x\}$ for $\{x\} \in h_i(\mathscr{B}_{i+1} \times \{0\})$ and $t \in I$.

Also, by the proof of (2.1) we see that

(8) $h'_{i+1}(A, t) \subset S(r(A), 2\sup\{\varrho(r(A), a) | \ a \in A\})$ for $A \in h_i(\mathscr{B}_{i+1} \times \{0\})$ and $t \in I$ (because in the proof of (2.1) f is a contraction).

Define a homotopy

$$h_{i+1}\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

by

$$h_{i+1}(A, t) = \begin{cases} h_i(A, 2t-1) & \text{if } A \in \omega^{-1}([0, t_0)) \text{ and } 1/2 \leqslant t \leqslant 1, \\ h'_{i+1}(h_i(A, 0), 2t + (1-2t)\varphi_{i+1}(A)) & \text{if } A \in \mathscr{B}_{i+1} \\ & \text{and } 0 \leqslant t \leqslant 1/2, \\ h_i(A, 0) & \text{if } A \text{ is not contained in } \mathscr{B}_{i+1} \text{ and } 0 \leqslant t \leqslant 1/2. \end{cases}$$

By A(i)–E(i), h_{i+1} satisfies the conditions A$(i+1)$–E$(i+1)$. By F(i) and (8), we see that h_{i+1} satisfies the conditions F$(i+1)$. Thus we have a homotopy

$$h' = h_n\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

such that

(9) $h'(A, 1) = A$, $h'(A, 0) = r(A) \in F_1(X)$ for $A \in \omega^{-1}([0, t_0))$;

(10) $h'(\{x\}, t) = \{x\}$ for $x \in X$ and $t \in I$;

(11) if $\omega(h'(A, s)) = \omega(h'(A, t))$ for some $A \in \omega^{-1}([0, t_0))$ and $s, t \in I$, then $h'(A, s) = h'(A, t)$.

By (9) and (11), we can define a function

$$h\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

by

(12) $h(A, t) = h'(A, \theta(A, t))$, where $\theta(A, t)$ is a positive number such that

$$\omega(h'(A, \theta(A, t))) = t \cdot \omega(A).$$

Then h is continuous. In fact, suppose, on the contrary, that there are a sequence $A_1, A_2, \ldots$ of points in $\omega^{-1}([0, t_0))$ and a sequence $t_1, t_2, \ldots$ of positive numbers in I such that

$$\lim A_n = A \in \omega^{-1}([0, t_0)) \quad \text{and} \quad \lim t_n = t \in I$$

and

$$\lim h(A_n, t_n) = B \neq h(A, t).$$

By (12),

$$\omega(B) = \lim(t_n \cdot \omega(A_n)) = t \cdot \omega(A) = \omega(h'(A, \theta(A, t))).$$

Note that $B \in h'(\{A\} \times I)$. Hence (11) implies that

$$B = h'(A, \theta(A, t)) = h(A, t).$$

This is a contradiction. Clearly, h satisfies the conditions (A1)′, (A2)′ and (A3)′. This completes the proof.

3. Whitney maps of certain polyhedra. In this section, we study Whitney maps of certain polyhedra. Let K be a cubical complex. Metrize $|K|$ as follows: Assume that each k-dimensional cube of K is a copy of I^k. Define the distance ϱ between two points x, y of $|K|$ so that if x, y are in a common cube I^k, then

$$\varrho(x, y) = \max\{|x_i - y_i| \mid i = 1, 2, \ldots, k\},$$
$$\text{where } x = (x_1, x_2, \ldots, x_k),\ y = (y_1, y_2, \ldots, y_k) \in I^k,$$

otherwise the distance is the length of the shortest path joining them (see [9]). Then ϱ is a convex metric. A connected subset Y of $|K|$ is GC (see [9]) if for any cube I^k of K either $Y \cap I^k = \emptyset$ or for some $0 \leqslant s_i \leqslant t_i \leqslant 1$ ($i = 1, 2, \ldots, k$)

$$Y \cap I^k = \{(y_1, y_2, \ldots, y_k) \in I^k \mid s_i \leqslant y_i \leqslant t_i \ (i = 1, 2, \ldots, k)\}.$$

A cubical complex K is *regular collapsible* [9] if there are a sequence of subcomplexes $K_0, K_1, \ldots, K_n$ of K and nonempty subcomplexes L_i of K_i such that K_0 is a one-point set, $K_n = K$ and

$$K_{i+1} = K_i \cup (L_i \times I),$$

where

$$L_i \times I = \{c \times \{0\}, c \times I, c \times \{1\} \mid c \in L_i\},$$

and each $|L_i|$ is GC of K_i. A cubical complex K is *locally regular collapsible* if for any $x \in |K|$ there is a regular collapsible subcomplex L of K such that $x \in \operatorname{Int}|L|$.

In [9], Mai and Tang proved that if P is a collapsible simplicial polyhedron, then there is a regular collapsible cubical complex K such that $|K| = P$. Also, they proved that the metric ϱ as above is a hyperconvex metric. Hence, by (2.1) and the proof of (1.2) (see [6]), we have

(3.1) THEOREM. *Let P be a finite collapsible polyhedron. Then there exists a strongly admissible Whitney map ω for $\mathscr{H} = 2^P$ or $C(P)$. Moreover, $\omega|\omega^{-1}((0, \omega(P)))$ is a trivial bundle map with Hilbert cube fibers, where if $\mathscr{H} = C(P)$, assume that P contains no free arc.*

Also, by (2.2) we have

(3.2) THEOREM. *Let K be a cubical complex and let $|K| = P$. If K is locally regular collapsible, then there is a Whitney map ω for $\mathscr{H} = 2^P$ or $C(P)$ such that for some $t_0 \in (0, \omega(P))$ there is a homotopy*

$$h\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

satisfying the conditions (A1)′, (A2)′ *and* (A3)′ *in* (2.2). *Moreover, $\omega|\omega^{-1}((0, t_0))$ is a trivial bundle map with $P \times Q$ fibers, where if $\mathscr{H} = C(P)$, assume that P contains no free arc.*

Next, we study Whitney maps of 1-dimensional ANRs.

(3.3) LEMMA. *If X is a compact 1-dimensional* AR (= *dendrite*), *then X admits a hyperconvex metric. If X is a compact connected 1-dimensional* ANR, *then X admits a locally hyperconvex metric.*

Proof. Suppose that X is a compact 1-dimensional AR. By (2.1) in [4] we can conclude that if any collection $\{A_i\}_{i=1,2,\dots,n}$ of subcontinua of X satisfies the condition $A_i \cap A_j \neq \emptyset$, then

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Since X is a Peano continuum, X admits a convex metric ϱ. Then, for any $x \in X$ and $\varepsilon > 0$, $S(x, \varepsilon)$ is a subcontinuum of X. Hence we see that ϱ is hyperconvex. Suppose that X is a compact 1-dimensional ANR. Let ϱ be a convex metric on X. By (13.6) in [2], there is a positive number $\varepsilon > 0$ such that $S(x, \varepsilon)$ is a 1-dimensional AR for each $x \in X$. Hence $\varrho|S(x, \varepsilon)$ is hyperconvex, which implies that ϱ is a locally hyperconvex metric.

(3.4) LEMMA. *If X_i $(i = 1, 2)$ admits a hyperconvex metric (resp., locally hyperconvex metric) ϱ_i, then $X_1 \times X_2$ admits a hyperconvex metric (resp., locally hyperconvex metric) ϱ, where*

$$\varrho(x, y) = \max\{\varrho_i(x_i, y_i) |\ x = (x_1, x_2),\ y = (y_1, y_2) \text{ and } i = 1, 2\}.$$

(3.5) COROLLARY. *Let X_i $(i = 1, 2, \dots, n)$ be the $m(i)$-sphere $(m(i) \geqslant 1)$ and let (X_{n+1}, ϱ_{n+1}) be a locally hyperconvex metric continuum. Suppose that*

$$X = \prod_{i=1}^{n+1} X_i.$$

Then there is a Whitney map ω for $\mathscr{H} = 2^X$ or $C(X)$ such that $\omega|\omega^{-1}((0, t_0))$ is a trivial bundle map with $X \times Q$ fibers for some $t_0 \in (0, \omega(X))$.

Outline of proof. Assume that

$$X_i = \{x = (x_0, x_1, \dots, x_{m(i)}) \in R^{m(i)+1} |\ \|x\| = 1\}.$$

We define a metric ϱ_i on X_i by

$$\varrho_i(x, y) = \arccos\Big[\sum_{j=0}^{m(i)} x_j y_j\Big]$$

for

$$x = (x_0, x_1, \ldots, x_{m(i)}),\ y = (y_0, y_1, \ldots, y_{m(i)}) \in X_i.$$

Define a metric ϱ on X by

$$\varrho(x, y) = \max\{\varrho_i(x_i, y_i) \mid i = 1, 2, \ldots, n+1\}$$

for

$$x = (x_1, x_2, \ldots, x_{n+1}),\ y = (y_1, y_2, \ldots, y_{n+1}) \in X.$$

Let ω be the Whitney map for $\mathscr{H}$ which is defined by (*) and the metric ϱ. By the similar way as in the proofs of (2.1) and (2.2), we can conclude that ω satisfies the desired conditions.

In the statements of (2.2), (3.2) and (3.5), we cannot conclude that $t_0 = \omega(X)$. We have the following

(3.6) PROPOSITION. *Let X be a compact connected* ANR *but not* AR. *Let $\mathscr{H} = 2^X$ or $C(X)$. If $\mathscr{H} = C(X)$, assume that X contains no free arc. Then for any Whitney map ω for $\mathscr{H}$ there is no homotopy*

$$h\colon \omega^{-1}([0, \omega(X))) \times I \to \omega^{-1}([0, \omega(X)))$$

satisfying the conditions (A1)′ *and* (A2)′ *in* (2.2).

Proof. Suppose, on the contrary, that such a homotopy h exists. Since $\mathscr{H}$ is homeomorphic to the Hilbert cube Q,

$$\omega^{-1}([0, \omega(X))) = Q - \{*\} = Q \times [0, 1).$$

Hence $\omega^{-1}([0, \omega(X)))$ is contractible. Let $f\colon \omega^{-1}([0, \omega(x))) \to F_1(X)$ $(= X)$ be a map defined by $f(A) = h(A, 0)$ for each $A \in \omega^{-1}([0, \omega(X)))$ (see (A1)′). Then (A2)′ implies that $f|F_1(X) \simeq 1_{F_1(X)}$. Since X is an ANR, by Borsuk's homotopy extension theorem (see [2]), there is a retraction

$$r\colon \omega^{-1}([0, \omega(X))) \to F_1(X).$$

Thus X is contractible, and hence X is an AR (see [2]). This is a contradiction.

(3.7) EXAMPLE. Let S^1 be the unit circle in the plane R^2 and let ϱ be the arc length metric on S^1. Suppose that ω is the Whitney map for 2^{S^1} defined by (*) and the metric ϱ. Then $\omega|\omega^{-1}((0, \pi/2))$ is a trivial bundle map with $S^1 \times Q$ fibers, but $\omega|\omega^{-1}((0, \pi/2])$ is not a trivial bundle map; in fact, it is not an open map. Let $A \in \omega^{-1}([0, \pi/2))$. First, we shall show that

($\#$) there are points $r_1(A)$ and $r_2(A)$ of A such that

$$\varrho(r_1(A), r_2(A)) < \pi \quad \text{and} \quad A \subset [r_1(A), r_2(A)],$$

where if $x, y \in S^1$, then

$$[x, y] = \{z \in S^1 | \ \varrho(x, z) + \varrho(z, y) = \varrho(x, y)\}.$$

If $|A| \leqslant 2$ (where $|A|$ denotes the cardinality of A), it is easily seen that ($\#$) is true. Let $|A| \geqslant 3$. Suppose, on the contrary, that ($\#$) is not true for some $A \in \omega^{-1}([0, \pi/2))$. Choose a point $a \in A$ and let a' be the point of S^1 such that $\varrho(a, a') = \pi$. Then A does not contain a'. Let S_a and $S_{a'}$ denote the path components of $S^1 - \{a, a'\}$. Choose the point $b \in A \cap S_a$ such that $A \cap S_a \subset [a, b]$. Then there is a point $c \in A \cap S_b$, where S_b does not contain a. Note that if $x \in \{a, b, c\}$, then $\{x, x'\}$ separates S^1 between the other two points $\{a, b, c\} - \{x\}$. Then we have

$$\omega(A) \geqslant \omega(\{a, b, c\}) = (1/2)\max\{\varrho(a, b), \varrho(b, c), \varrho(c, a)\}$$
$$+ (1/4)\min\{\varrho(a, b), \varrho(b, c), \varrho(c, a)\} \geqslant \pi/2.$$

This is a contradiction. By ($\#$) we can easily see that there is a homotopy

$$h\colon \omega^{-1}([0, \pi/2)) \times I \to \omega^{-1}([0, \pi/2))$$

satisfying the conditions (A1)′, (A2)′ and (A3)′ in (2.2). Hence $\omega|\omega^{-1}((0, \pi/2))$ is a trivial bundle map with $S^1 \times Q$ fibers. Let

$$x_1 = (1, 0), \quad x_2 = (-1/2, \sqrt{3}/2), \quad x_3 = (-1/2, -\sqrt{3}/2)$$

and let $A = \{x_1, x_2, x_3\}$. Then $\omega(A) = \pi/2$. In [5], (4.15), Goodykoontz and Nadler pointed out the following fact: there is a neighborhood $\mathscr{U}$ of A in 2^{S^1} such that if $B \in \mathscr{U}$, then $\omega(B) \geqslant \pi/2$. This implies that $\omega|\omega^{-1}([0, \pi/2])$ is not an open map.

The following problems remain open:

(i) Let X be a compact AR. Is there a strongly admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$? (see [6], (3.4)). **(P 1358)**

(ii) Let X be a compact ANR (or polyhedron). Is there a Whitney map ω for $\mathscr{H} = 2^X$ or $C(X)$ such that for some $t_0 \in (0, \omega(X))$ there is a homotopy

$$h\colon \omega^{-1}([0, t_0)) \times I \to \omega^{-1}([0, t_0))$$

satisfying the conditions (A1)′, (A2)′ and (A3)′ in (2.2)? **(P 1359)**

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