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# SETS OF INTERPOLATION FOR FOURIER TRANSFORMS OF BIMEASURES

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**1. Introduction.** Let $X$ and $Y$ be locally compact spaces. We consider the space of all bounded, bilinear forms on $C_0(X) \times C_0(Y)$. In other words, if $V_0(X, Y)$ denotes the projective tensor product $C_0(X) \hat{\otimes} C_0(Y)$, then we are concerned with elements of the dual space $BM(X, Y) = V_0(X, Y)^*$. $V_0(X, Y)$ is called the *Varopoulos algebra* on $X \times Y$, and the elements of $BM(X, Y)$ are referred to as bimeasures.

Suppose now that we are given locally compact, abelian groups $G$ and $H$. In [4] we began the study of the harmonic analysis of bimeasures on $G \times H$. We showed that $BM(G, H)$ is a Banach algebra under convolution, in appropriate norm, and we studied its algebra structure. Further investigations of bimeasure algebras were carred out in [1], where the context is expanded to include nonabelian groups $G$ and $H$, and in [2], where multilinear forms in any number of variables are considered.

Let $\hat{G}$ and $\hat{H}$ be the character groups of $G$ and $H$, respectively. If $u \in BM(G, H)$, then the Fourier transform $\hat{u}$ of $u$ is a well-defined, bounded, uniformly continuous function on $\hat{G} \times \hat{H}$, and the Fourier transform is a norm-decreasing, injective, algebra homomorphism from $BM(G, H)$ into the algebra $C_b(\hat{G} \times \hat{H})$ of bounded, continuous functions on $\hat{G} \times \hat{H}$ ([4], Defs. 1.10, 2.5); cf. ([1], Def. 1.2ff.). In fact, the Fourier transforms of bimeasures on $G \times H$ can be characterized as follows. A function $\alpha$ on $\hat{G} \times \hat{H}$ is the Fourier transform of an element of $BM(G, H)$ if and only if there exist strongly continuous, unitary representations $\pi$ of $\hat{G}$ and $\sigma$ of $\hat{H}$ on a Hilbert space $\mathscr{H}$ and vectors $\xi, \eta \in \mathscr{H}$ such that

$$\alpha(\gamma, \chi) = \langle \pi(\gamma)\xi, \sigma(\chi)\eta \rangle, \quad \gamma \in \hat{G}, \ \chi \in \hat{H}.$$

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Our main purpose in the present note is to pursue the study of sets of interpolation for the algebra $BM^{\wedge}(\hat{G}, \hat{H})$ of Fourier transforms of bimeasures in $BM(G, H)$, begun in ([4], Sec. 6). Thus a subset $E$ of $\hat{G} \times \hat{H}$ will be called a *$BM$-interpolation set* if for every bounded, continuous function $f$ on $E$ there exists $u \in BM(G, H)$ such that $u|E = f$. (Such sets were dubbed "$BM$-Sidon" sets in [4].) Of course, in that case there is a constant $\alpha(E)$ such that $u$ can be chosen as above with $\|u\| \leqslant \alpha(E) \|f\|_E$, where $\|f\|_E$ denotes the norm of $f$ as an element of $C_b(E)$.

We shall begin our study of $BM$-interpolation sets by relating $BM(G, H)$ to the multiplier algebra $N(\hat{G}, \hat{H})$ and the tilde algebra $\tilde{V}_0(\hat{G}, \hat{H})$ of $V_0(\hat{G}, \hat{H})$ and recalling known characterizations of sets of interpolation for these spaces. If $X$ and $Y$ are locally compact spaces, then a function $f \in C_b(X \times Y)$ belongs to $N(X, Y)$ if $fg \in V_0(X, Y)$ for all $g \in V_0(X, Y)$, while to say that $f \in \tilde{V}_0(X, Y)$ is to say that there is a bounded net $\{f_\alpha\}$ in $V_0(X, Y)$ converging to $f$ uniformly on compact sets. The norm on $N(X, Y)$ is the usual operator norm, and $\tilde{V}_0(X, Y)$ is normed by taking $\|f\|$ to be the infimum over all nets $\{f_\alpha\}$ (as above) of the numbers $\sup_\alpha \|f_\alpha\|$. Of course, if $X$ and $Y$ are both compact, then

$$N(X, Y) = V_0(X, Y) \subseteq \tilde{V}_0(X, Y);$$

in general the inclusion on the right is proper. For any $X$ and $Y$, set

$$V(X, Y) = C_b(X) \hat{\otimes} C_b(Y)$$

and let $\tilde{V}(X, Y) = \tilde{V}_0(\beta X, \beta Y)$ denote the space of uniform limits on $X \times Y$ of bounded sequences in $V(X, Y)$. Then we have

$$V(X, Y) \subset N(X, Y) \subset \tilde{V}_0(X, Y).$$

(To prove the inclusion on the right, apply [6], ([5], Thm. 4.5); cf. the proof of Theorem 1 below.) In fact, $N(X, Y)$ consists precisely of the functions $f$ in $\tilde{V}_0(X, Y)$ satisfying $f|E \times F \in V_0(E, F)$ for all compact sets $E \subset X$ and $F \subset Y$. In particular, if $X$ and $Y$ are both discrete, then $N(X, Y) = \tilde{V}(X, Y)$ ([6], Thm. 3), ([7], Sec. 1).

**2. Bimeasures, tilde algebras and interpolation.** Recall that if $A(\hat{G} \times \hat{H})$ and $B(\hat{G} \times \hat{H})$ denote, as is customary, the Fourier algebra and the Fourier–Stieltjes algebra on $\hat{G} \times \hat{H}$, then

$$A(\hat{G} \times \hat{H}) \subset V_0(\hat{G}, \hat{H})$$

and

$$B(\hat{G} \times \hat{H}) \subset N(\hat{G}, \hat{H})$$

([4], Remark 1.11(1)). We now extend this sequence of inclusions by showing that

$$BM\hat{}(\hat{G}, \hat{H}) \subset \tilde{V}_0(\hat{G}, \hat{H}).$$

THEOREM 1. *Let G and H be locally compact, abelian groups with character groups $\hat{G}$ and $\hat{H}$, respectively. If $u \in BM(G, H)$, then $\hat{u} \in \tilde{V}_0(\hat{G}, \hat{H})$ and*

$$\|\hat{u}\|_{\tilde{V}_0} \leqslant \|u\|.$$

Proof. Let $u \in BM(G, H)$. By ([5], Thm. 4.5), the conclusions of our theorem are equivalent to the assertion that for every bimeasure (measure) $v \in BM(\hat{G}, \hat{H})$ with finite support

$$|\langle \hat{u}, v\rangle| \leqslant \|u\| \, \|v\|.$$

Moreover, it follows from ([4], Lemma 1.13) that it suffices to assume that $u$ also has finite support in $G \times H$. Thus we shall assume $u$ and $v$ have finite support and write

$$\langle f, u\rangle = \sum_{i,j=1}^{m} a_{ij} f(x_i, y_j), \quad f \in V_0(G, H)$$

and

$$\langle \phi, v\rangle = \sum_{k,l=1}^{n} b_{kl} \phi(\gamma_k, \chi_l), \quad \phi \in V_0(\hat{G}, \hat{H}).$$

Let $u \otimes v$ denote the element of $BM(G \times \hat{G}, H \times \hat{H})$ whose existence is asserted in ([1], Thm. 2.1), and set

$$f(x, \gamma) = \langle -x, \gamma\rangle, \quad x \in G, \ \gamma \in \hat{G}$$

and

$$g(y, \chi) = \langle -y, \chi\rangle, \quad y \in H, \ \chi \in \hat{H}.$$

Since $u$ is finitely supported, $\hat{u} \in V(\hat{G}, \hat{H})$, and

$$\begin{aligned}\langle \hat{u}, v\rangle &= \sum_{k,l=1}^{n} b_{kl} \hat{u}(\gamma_k, \chi_l) \\ &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{m} a_{ij} b_{kl} \langle -x_i, \gamma_k\rangle \langle -y_j, \chi_l\rangle \\ &= \langle f \otimes g, u \otimes v\rangle.\end{aligned}$$

Thus [1, Thm. 2.1] gives

$$|\langle \hat{u}, v\rangle| \leqslant \|u \otimes v\| \, \|f\| \, \|g\| \leqslant \|u\| \, \|v\|,$$

as desired.

It follows from Theorem 1 that every $BM$-interpolation set in $\hat{G}\times\hat{H}$ is a $\tilde{V}(\hat{G},\hat{H})$-interpolation set. In particular, if $G$ and $H$ are compact, then $BM$-interpolation sets are sets of interpolation for the algebra $N(G, H)$. The structure of the latter is known [7] and leads to Corollary 2 below.

Following [7], if $X$ and $Y$ are discrete spaces, then a subset $E \subset X\times Y$ is called a 1-*section* if

$$\operatorname{card}(p_X^{-1}(x)\cap E)\leqslant 1$$

for all $x\in X$, where $p_X$ is the projection of $X\times Y$ onto $X$, and a 2-section is defined analogously in terms of $p_Y$. If $E$ is either a 1- or 2-section, we shall call it a *section.*

COROLLARY 2. *Let $G$ and $H$ be discrete abelian groups, and let $E\subset\hat{G}\times\hat{H}$ be a $BM$-interpolation set.*

(i) *$E$ is a finite union of sections.*

(ii) *$p_{\hat{H}}(p_{\hat{G}}^{-1}(\gamma)\cap E)$ and $p_{\hat{G}}(p_{\hat{H}}^{-1}(\chi)\cap E)$ are Sidon sets in $\hat{H}$ and $\hat{G}$, respectively, for all $\gamma\in\hat{G}$ and $\chi\in\hat{H}$.*

(iii) $\sup_{\gamma\in\hat{G}}\alpha(p_{\hat{G}}^{-1}(\gamma)\cap E)<\infty$ *and* $\sup_{\chi\in\hat{H}}\alpha(p_{\hat{H}}^{-1}(\chi)\cap E)<\infty.$

Proof. By ([7], Thm. 4.1 (i)) $E$ is a $V$-Sidon set, that is every function in $C_0(E)$ can be extended to an element of $V_0(\hat{G},\hat{H})$. By ([7], Thms. 4.2, 4.3) every $V$-Sidon set is a finite union of sections. The rest of the Corollary follows from the fact that if $\gamma\in\hat{G}$, $\chi\in\hat{H}$ and $u\in BM(G, H)$, then the functions $\gamma\to\hat{u}(\gamma,\chi)$ and $\chi\to\hat{u}(\gamma,\chi)$ are Fourier–Stieltjes transforms on $\hat{G}$ and $\hat{H}$, respectively.

Remarks. (1) Note that the quantities $\alpha$ appearing in (iii) are the classical Sidon constants of the corresponding Sidon sets appearing in (ii).

(2) It is natural to ask whether conditions (i)–(iii) of Corollary 2 characterize $BM$-interpolation sets (**P 1294**). Although we do not know whether this is the case in general, a partial converse to Corollary 2 is valid. Namely, it follows from trivial modifications of ([4], Thm. 6.3), replacing $p_1$ by a subset thereof, and ([4], Cor. 6.7) that of $E$ is a finite union of sections in $G\times H$, such that (iii) holds, with "$\alpha$" replaced by "card", then $E$ is a $BM$-interpolation set.

**3.** ***BM/B* sets.** There is a notion which is, in a natural sense, related to the concept of $BM$-interpolation set. For any subset $E$ of $\hat{G}\times\hat{H}$, let $B(E)$, $BM\hat{\ }(E)$ and $N(E)$ denote the corresponding algebras of restrictions to $E$, so that Theorem 1 gives

$$B(E)\subseteq BM\hat{\ }(E)\subseteq N(E).$$

We shall call $E$ a *$BM/B$ set* if $BM\hat{\ }(E)=B(E)$; that is, $E$ is a $BM/B$ set if the only functions on $E$ which can be extended to elements of $BM\hat{\ }(G, H)$ are the obvious ones, namely the restrictions of Fourier–Stieltjes transforms.

We are unable to give a strong necessary condition for a set $E$ to be a $BM/B$ set at present. Here are two examples that illustrate the question.

PROPOSITION 3. *Let $E$ and $F$ be disjoint subsets of the discrete group $\hat{G}$ such that $E \cup F$ is a $K$-set in the sense of* ([7], Sec. 3). *Then $E \times F$ is a $BM/B$ set in $\hat{G} \times \hat{G}$.*

Proof. By ([7], Thm. 3.1) we have

$$N(E \times F) \subset B(E \times F) \subset BM\hat{\ }(E \times F) \subset N(E \times F),$$

so

$$B(E \times F) = BM\hat{\ }(E \times F).$$

Remark. $E \times F$ is not a $BM$-interpolation set, since $N(E, F) = N(E \times F) \neq C_b(E \times F)$ [7].

PROPOSITION 4. *Let $E$ be an infinite subset of the discrete group $G$, and let $H$ be an infinite discrete abelian group. Then $E \times H$ is not a $BM/B$ set.*

Proof. It is easy to see that a set $F$ is a $BM/B$ set if and only if there is a constant $C > 0$ such that $\|f\|_V \leqslant C\|f\|_\infty$ for all trigonometric polynomials $f$ with frequencies (only) in $F$. We shall construct a sequence of polynomials with frequencies in $E \times H$ such that $\|f\|_V / \|f\|_\infty \to \infty$.

Fix $n \geqslant 1$. Let $U$ be a neighbourhood of $0 \in H$ and $x_1, \ldots, x_n \in H$ such that $(x_j + U) \cap (x_k + U) = \emptyset$ for $1 \leqslant j \neq k \leqslant n$.

Let $g$ be a trigonometric polynomial on $H$ such that $\|g\|_\infty = g(0) = 1$ and $|g| \leqslant 1/n^4$ outside of $U$.

Our polynomial $f$ will be of the form

$$f = \sum_1^n \gamma_k \times (\delta(-x_k) * g),$$

where the $\gamma_j$ remain to be chosen.

We shall also choose $\varepsilon_{i,j} = \pm 1$, $1 \leqslant i, j \leqslant n$, and $y_1, \ldots, y_n \in G$ such that for $u = \sum_{i,j} \varepsilon_{i,j} \delta(y_i) \times \delta(x_j)$, $\|u\|_{BM} \leqslant Cn^{3/2}$, where $C$ is a constant that does not depend on $n$. (The existence of $C$ and $\varepsilon_{i,j}$ is guaranteed by [3, 11.8].) If the $\gamma_k$ and $y_i$ are such that $\operatorname{Re} \langle \gamma_k, y_j \rangle \varepsilon_{j,k} \geqslant 1/4$ for all $j$, $k$, then

$$\begin{aligned}
|\langle u, f\rangle| &= \Big|\sum_{i,j,k} \varepsilon_{i,j} \langle \delta(y_i) \times \delta(x_j), \gamma_k \times \delta(-x_k) g(0)\rangle\Big| \\
&\geqslant \Big|\sum_{i,j} \varepsilon_{i,j} \langle y_i, \gamma_j\rangle g(0)\Big| - \Big|\sum_{\substack{i,j \\ k \neq j}} \varepsilon_{i,j} \langle y_i, \gamma_k\rangle g(x_j - x_k)\Big| \\
&\geqslant n^2/4 - n^2(n-1)/n^4 \geqslant n^2/8.
\end{aligned}$$

It follows that $\|f\|_V \geqslant (n^2/8)/Cn^{3/2} \geqslant C' n^{1/2}$. Of course $\|f\|_\infty \leqslant 1 + (n-1)/n^4$. Thus, $\|f\|_V / \|f\|_\infty \to \infty$.

It remains to choose the $\gamma_j$ and $y_k$, which is easily done. We may pass to a subsequence $\{\gamma_j\}$ of $E$ such that for every function $h$: $E \to \{\pm 1\}$, there exists $y \in G$ such that $\mathrm{Re}\big(h(\gamma)\langle y, \gamma\rangle\big) > 1/4$. That is a standard ($\varepsilon$-Kronecker set) construction, and we omit the details.

Remark. As of this writing we do not know whether $E \times (F_1 + F_2)$ is a $BM/B$ set when $E$ and $F_1 \cup F_2$ are lacunary.

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