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ON FILLING AN IRREDUCIBLE CONTINUUM WITH CHAINABLE CONTINUA

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Let $\mathscr{K}$ denote the class of continua to which K belongs if and only if there exist an irreducible continuum M and an upper semicontinuous decomposition G of M such that M/G is an arc and each element of G is homeomorphic to K. In this paper, *continuum* means a compact connected metric space. In [2] the question of whether each element of $\mathscr{K}$ contains two disjoint copies of itself is raised. In this paper it is shown by examples the answer to that question is negative. Indeed, elements of $\mathscr{K}$ need not even properly contain copies of themselves.

THEOREM 1. *Suppose K is a chainable continuum and that if a is an arc not intersecting K, then there is an arc a' intersecting $a\cup K$ at two points such that $a\cup a'\cup K$ is homeomorphic to K. Then K is in $\mathscr{K}$.*

Proof. Let C denote a Cantor set on $[0,1]$, $c_1, c_2, \ldots$ denote the left endpoints of the components of the complement of C on $[0,1]$, and $d_1, d_2, \ldots$ the right endpoints. For each positive integer i, let f_i denote a $(1/i)$-map from K onto $[0,1]$. Let G_1 denote the collection to which g belongs if and only if one of the following conditions holds:

(1) for some point P of $C-\bigcup_{i=1}^{\infty}\{c_i\}$ and some point Q of K, g is $P\times Q$;

(2) for some point z of $[0,1]$ and some positive integer i, g is $c_i\times f_i^{-1}(z)$.

If $g_1, g_2, \ldots$ are elements of G_1 belonging to different components of $C\times K$, then

$$(*)\qquad \lim_{i\to\infty} d(g_i) = 0.$$

Therefore, G_1 is an upper semicontinuous collection of mutually exclusive closed point sets filling up $C\times K$. For each i, $(c_i\times K)/G_1$ is an arc a_i. Then, by the hypothesis, for each i there is an arc a_i' such that $a_i\cup a_i'\cup(d_i\times K)$ is homeomorphic to K and

$$\overline{\bigcup_{i=1}^{\infty} a_i'} = (C\times K)/G_1\cup\bigcup_{i=1}^{\infty} a_i'.$$

Let

$$M = (C\times K)/G_1 \cup \bigcup_{i=1}^{\infty} a_i'.$$

Let G denote the collection to which g belongs if and only if one of the following conditions holds:

(1) for some point c of $C-\bigcup_{i=1}^{\infty}\{c_i, d_i\}$, g is $c\times K$;

(2) for some positive integer i, g is $a_i\cup a_i'\cup(d_i\times K)$.

Each element of G is homeomorphic to K and M/G is an arc. M is irreducible since if D is an open set containing a point of M, then D contains a point of $\bigcup_{i=1}^{\infty} a_i'$ and, therefore, a separating point of M.

THEOREM 2. *Suppose K is a chainable indecomposable continuum and there exists a sequence $K_1, K_2, \ldots$ of proper subcontinua of K such that*

(1) $\overline{\bigcup_{i=1}^{\infty} K_i} = K$,

(2) *if i and j are positive integers, then K_i and K_j do not belong to the same composant of K,*

(3) *for each positive integer i, there is a continuous map f_i from K onto K_i such that if P is a point of K, then the distance from P to $f_i(P)$ is less than $1/i$.*

Then K is in $\mathscr{K}$.

Proof. Let C, $\{c_i\}$, and $\{d_i\}$ be defined as in Theorem 1. Let G_1 denote the collection to which g belongs if and only if one of the following conditions holds:

(1) for some c in $C-\bigcup_{i=1}^{\infty} c_i$ and some point P of K, g is $c\times P$;

(2) for some positive integer i and some point P of K, g is $c_i\times f_i^{-1}(P)$.

As before, G_1 is an upper semicontinuous collection of mutually exclusive continua filling up $C\times K$ since if $g_1, g_2, \ldots$ are elements of G_1 in different components of $C\times K$, then (*) holds.

Let G_2 denote the collection to which g belongs if and only if one of the following conditions holds:

(1) for some point c of $C-\bigcup_{i=1}^{\infty}\{c_i, d_i\}$ and some point P of K, g is $c\times P$;

(2) for some positive integer i and some point P of K, g is $(P\times d_i)\cup(f_i^{-1}(P)\times c_i)$.

Again, G_2 is an upper semicontinuous collection of mutually exclusive closed point sets filling up $C\times K$ since $\lim_{i\to\infty} f_i$ is the identity, and hence if $g_1, g_2, \ldots$ are elements of G_2 in different components of $C\times K$, then (*) holds.

Let M denote $(C \times K)/G_1/G_2$.

Suppose M is not irreducible. Then there is a proper subcontinuum M' of M intersecting $(0 \times K)/G_1/G_2$ and $(1 \times K)/G_1/G_2$. There exist a point P of K, a neighborhood D_P of P, and integers i_1 and i_4 such that the set

$$M'' = \{[c_{i_1}, c_{i_4}] \cap C \times D_P\}/G_1/G_2$$

is a subset of $M - M'$. If Q_1 and Q_2 are points of $K - D_P$ lying in different composants of K, then Q_1 and Q_2 are not in the same component of $K - D_P$. Suppose c is in the common part of the open interval (c_{i_1}, c_{i_4}) with C, and K' is a component of $K - D_P$. Then there are integers i_2 and i_3 such that $c_{i_1} < c_{i_2} < c < c_{i_3} < c_{i_4}$, and K_{i_2} and K_{i_3} do not belong to the same composant of K which contains K'. The set $(c \times K')/G_1/G_2$ is a component of

$$M - [\{(c_{i_2} \times K_{i_2}) \cup (c_{i_3} \times K_{i_3})\}/G_1/G_2 \cup M''].$$

Therefore, M' does not intersect $(c \times K)/G_1/G_2$. But $(c \times K)/G_1/G_2$ separates $(0 \times K)/G_1/G_2$ from $(1 \times K)/G_1/G_2$ in M. This involves a contradiction. Therefore, M is irreducible.

Let G denote the collection to which g belongs if and only if one of the following conditions holds:

(1) for some c of $C - \bigcup_{i=1}^{\infty} \{c_i, d_i\}$, g is $c \times K$;

(2) for some i, g is $[(c_i \cup d_i) \times K]/G_1/G_2$.

Each element of G is homeomorphic to K and M/G is an arc.

THEOREM 3. *Suppose K is a chainable indecomposable continuum and there exist two composants K' and K'' of K and two sequences of compact subcontinua $h_1, h_2, \ldots$ and $k_1, k_2, \ldots$ of K' and K'', respectively, such that for each positive integer i there are maps f_i and g_i from M onto h_i and k_i, respectively, such that*

(1) *if P is in K, then the distances from P to $f_i(P)$ and from P to $g_i(P)$ are less than $1/i$;*

(2) $\overline{\bigcup_{i=1}^{\infty} h_i} = \overline{\bigcup_{i=1}^{\infty} k_i} = K.$

Then K is in $\mathscr{K}$.

The proof of this theorem is analogous to that of Theorem 2.

THEOREM 4. *Suppose K is a chainable indecomposable continuum and there exist a composant K' of K and a sequence $K_1, K_2, \ldots$ of mutually exclusive subcontinua of K' such that*

$$\overline{\bigcup_{i=1}^{\infty} K_i} = K$$

and for each positive integer i there is a map f_i from K onto K_i such tha if P is a point of K, then the distance from P to $f_i(P)$ is less than $1/i$. Then K is in $\mathscr{K}$.

Proof. Let C, $\{c_i\}$, and $\{d_i\}$ be defined as in the proof of Theorem 2. Also, let G_1, G_2, G, and M be defined as in the proof of Theorem 2 with respect to the sequence $K_1, K_2, \ldots$ of the hypothesis of this theorem.

Let us suppose again that M is not irreducible and let M', P, D_P, and i_1, i_4 have the same meaning as in the quoted proof. If Q_1 and Q_2 are points of $K-D_P$ lying in different composants of K, then Q_1 and Q_2 are in different components of $K-D_P$. Suppose c is in the common part of the open interval (c_{i_1}, c_{i_4}) with C, and N is a component of $K-D_P$ not contained by the composant K' of K. Then there are integers i_2 and i_3 such that $c_{i_1} < c_{i_2} < c < c_{i_3} < c_{i_4}$, and $(c \times N)/G_1/G_2$ is a component of

$$M-[\{(c_{i_2} \times K_{i_2}) \cup (c_{i_3} \times K_{i_3})\}/G_1/G_2 \cup M''].$$

Therefore, M' does not intersect $[c \times (K-K')]/G_1/G_2$.

There exist integers j_1, j_2, and j_3 such that, for each $n=1,2,3$, $(c_{j_n} \times K_{j_n})/G_1/G_2$ intersects M''. Since $\{[c_{i_1}, c_{i_4}] \cap C \times K\}/G_1/G_2 \cap M'$ is connected, there is a component E of $K-D_P$ such that $\{[c_{i_1}, c_{i_4}] \cap C \times E\}/G_1/G_2$ contains all the points of M' that are in $\{[c_{i_1}, c_{i_4}] \cap C \times K\}/G_1/G_2$. Then E must intersect K_{j_1}, K_{j_2}, and K_{j_3}. The continuum $E \cup K_{j_1} \cup K_{j_2} \cup K_{j_3}$ is a triod; but K is chainable. This involves a contradiction. Therefore, M is irreducible.

As before, M/G is an arc and each element of G is homeomorphic to K. Therefore, K is in $\mathscr{K}$.

Obvious examples of chainable continua described in these theorems are the curve $\sin 1/x$ and the indecomposable continuum due to Knaster (see [1], p. 209).

REFERENCES

[1] C. Kuratowski, *Théorie de continus irréductibles entre deux points. I*, Fundamenta Mathematicae 3 (1922), p. 200-231.

[2] W. R. R. Transue, J. W. Hinrichsen and B. Fitzpatrick, Jr., *Concerning upper semicontinuous decompositions of irreducible continua*, Proceedings of the American Mathematical Society 30 (1971), p. 157-163.

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