COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

ON EXACT MODULES OVER A COMMUTATIVE EXACT RING

BY

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Introduction. Let R be a commutative ring with identity and let M be an R-module. Let (u_1, v_1) be a pair of elements in R. We say that M is (u_1, v_1)-*exact* if the sequence

$$M \xrightarrow{u_1} M \xrightarrow{v_1} M \xrightarrow{u_1} M$$

is exact. Let (u, v) be a sequence of pairs $(u_1, v_1), \ldots, (u_n, v_n)$. Then M is (u, v)-*exact* if M_i is (u_i, v_i)-exact for $i = 1, 2, \ldots, n$, where $M_1 = M$, $M_i = M/(u_1, \ldots, u_{i-1})M$. The category of all R-modules we denote by R-Mod.

In this paper, for a given sequence (u, v) of pairs in R the full subcategory of R-Mod consisting of all (u, v)-exact R-modules is investigated under the assumption that R is (u, v)-exact.

For basic examples and preliminary results the reader is referred to [2].

In Section 1 a homological characterization of (u, v)-modules is given. We obtain a completion of Lemma 1.4 in [2] and a generalization of Theorem 1.5 in [2].

In Section 2 we study the structure of (u, v)-exact modules over an $\aleph_0$-Noetherian ring. Following some ideas of Simson [3]-[5] we prove that every such module is a directed union of $\aleph_0$-generated (u, v)-exact submodules.

1. A homological characterization of (u, v)-exact modules. We begin with the following result:

LEMMA 1. *Let R be (u_1, v_1)-exact and let M be an R-module. If P is a projective generator in R-Mod and I is an injective cogenerator in R-Mod, then for $M_1 = M/u_1M$ and $R_1 = R/u_1R$ the following conditions are equivalent*:

(1) *M is (u_1, v_1)-exact.*

(2) $\mathrm{Tor}_n^R(R_1, M) = 0$ *for* $n \geqslant 1$.

(3) $\mathrm{Ext}_R^n(R_1, M) = 0$ *for* $n \geqslant 1$.

(4) $\mathrm{Tor}_n^R(N, M) \simeq \mathrm{Tor}_n^{R_1}(N, M_1)$ *for* $n \geqslant 1$ *and any R_1-module N.*

(5) $\mathrm{Ext}_R^n(N, M) \simeq \mathrm{Ext}_{R_1}^n(N, M_1)$ *for* $n \geqslant 1$ *and any R_1-module N.*

(6) $\mathrm{Ext}_R^n(M, I_1) = 0$ *for* $n \geqslant 1$.

(7) $\mathrm{Ext}_R^n(P_1, M) = 0$ *for* $n \geqslant 1$.

(8) $\mathrm{Ext}_{R_1}^n(M_1, N) \simeq \mathrm{Ext}_R^n(M, N)$ *for* $n \geqslant 1$ *and any R_1-module N.*

Proof. The equivalence of conditions (1)-(5) is proved in [2], Lemma 1.4. It follows that for the injective cogenerator I the complex

$$\boldsymbol{I} : I \xrightarrow{u_1} I \xrightarrow{v_1} I \xrightarrow{u_1} I \longrightarrow \ldots$$

is exact and, therefore, it is an injective resolution of $I_1 \simeq v_1 I$. Hence

$$\mathrm{Ext}_R^n(M, I_1) \simeq H^n(\mathrm{Hom}_R(M, \boldsymbol{I})).$$

(1) $\Leftrightarrow$ (6). If M is (u_1, v_1)-exact and I is the injective cogenerator, then the complex $\mathrm{Hom}_R(M, \boldsymbol{I})$ is acyclic and $\mathrm{Ext}_R^n(M, I_1) = 0$ for $n \geqslant 1$. Conversely, suppose that $\mathrm{Ext}_R^n(M, I_1) = 0$ for $n \geqslant 1$. Then the exact sequence

$$(*) \qquad \ldots \longrightarrow M \xrightarrow{u_1} M \xrightarrow{v_1} M \xrightarrow{u_1} M \longrightarrow \ldots$$

induces the sequence

$$\ldots \longrightarrow \mathrm{Hom}_R(M, I) \xrightarrow{u_1} \mathrm{Hom}_R(M, I) \xrightarrow{v_1} \mathrm{Hom}_R(M, I) \xrightarrow{u_1} \ldots,$$

which is exact because it equals $\mathrm{Hom}_R(M, \boldsymbol{I})$. Since I is an injective cogenerator, sequence $(*)$ is exact.

(8) $\Rightarrow$ (6) is clear since, by (1) $\Leftrightarrow$ (5), I_1 is R_1-injective.

(1) $\Rightarrow$ (8). Let M be a (u_1, v_1)-exact module and let N be an R_1-module. If $\boldsymbol{P}$ is a projective resolution of M, then it follows from (5) that $\boldsymbol{P}_1 = \boldsymbol{P} \otimes_R R_1$ is a projective resolution of the R_1-module M_1. Then (8) follows since there are isomorphisms

$$\mathrm{Hom}_R(\boldsymbol{P}, N) \simeq \mathrm{Hom}_{R_1}(\boldsymbol{P}_1, N) \quad \text{and} \quad \mathrm{Ext}_{R_1}^n(M_1, N) \simeq \mathrm{Ext}_R^n(M, N) \quad \text{for } n \geqslant 1.$$

Since equivalence (7) $\Leftrightarrow$ (6) can be proved similarly, the proof of the lemma is complete.

Now we prove the following characterization of exact modules.

THEOREM 1. *Let (u, v) be a sequence of pairs $(u_1, v_1), \ldots, (u_n, v_n)$ in R. If R is (u, v)-exact and M is an R-module, then the following conditions are equivalent:*

(1) *M is (u, v)-exact.*

(2) $\operatorname{Ext}_R^m(P_i, M) = 0$ *for* $m > 0$, $i = 1, 2, \ldots, n$ *and a projective generator P in R*-Mod.

(3) $\operatorname{Ext}_R^m(M, I_i) = 0$ *for* $m > 0$, $i = 1, 2, \ldots, n$ *and an injective cogenerator I in R*-Mod.

Proof. Equivalence (1)$\Leftrightarrow$(2) can be proved as equivalence (1)$\Leftrightarrow$(3) in Theorem 1.5 in [2].

(1)$\Leftarrow$(3). Suppose that (3) is satisfied and let I be an injective cogenerator in R-Mod. Since $\operatorname{Ext}_R^m(M, I_1) = 0$, M is (u_1, v_1)-exact by Lemma 1. Suppose that M is $((u_1, v_1), \ldots, (u_{k-1}, v_{k-1}))$-exact for a certain k $(1 \leqslant k < n)$. By assumption and (8) in Lemma 1,

$$\operatorname{Ext}_{R_i}^m(M_i^!, N) \simeq \operatorname{Ext}_R^m(M, N) \qquad \text{for all } 1 \leqslant i < k$$

whence

$$0 = \operatorname{Ext}_R^m(M, I_{k+1}) \simeq \ldots \simeq \operatorname{Ext}_{R_k}^m(M_k, I_{k+1}).$$

Then by Lemma 1 again, M_k is a (u_k, v_k)-exact module.

(1)$\Rightarrow$(3). Let M be (u, v)-exact and let I be an injective cogenerator in R-Mod. By Lemma 1 we have

$$\operatorname{Ext}_R^m(M, I_i) \simeq \operatorname{Ext}_{R_i}^m(M_i, I_i) = 0,$$

since I_i is R_i-injective.

2. Structure of exact modules. We recall that a ring is $\aleph_0$-*Noetherian* if each of its ideals is generated by at most $\aleph_0$-elements [1].

LEMMA 2. *Let R be an $\aleph_0$-Noetherian ring and let M be an $\aleph_0$-generated R-module. Then every submodule of M is countably generated.*

The lemma follows from Theorem 1 in [1].

The main result of this section is the following

THEOREM 2. *Let (u, v) be a sequence of pairs $(u_1, v_1), \ldots, (u_n, v_n)$ in R and let R be an $\aleph_0$-Noetherian (u, v)-exact ring. Then any (u, v)-exact R-module is a directed union of $\aleph_0$-generated (u, v)-exact submodules.*

To prove the theorem we need some technical lemmas.

LEMMA 3. *Let R be an $\aleph_0$-Noetherian (u_1, v_1)-exact ring and let M be a (u_1, v_1)-exact R-module. Then every countably generated R-submodule K of M can be embedded into a countably generated (u_1, v_1)-exact submodule $\bar{K}$ of M.*

Proof. Let K be a submodule of M generated by elements m_i, $i \in N$, where N is the set of natural numbers. We now consider the complex of R-modules

$$M(u_1, v_1)\colon \longrightarrow M \xrightarrow{u_1} M \xrightarrow{v_1} M \xrightarrow{u_1} M \longrightarrow \ldots$$

and its subcomplex $K(u_1, v_1)$. By Lemma 2, the R-module $\mathrm{Ker}(u_1|K)$ is countably generated. Let $\{x_k^{(1)}\}_{k\in N}$ be a set of its generators. Since $u_1 x_k^{(1)} = 0$ for each $k \in N$, there exist elements $y_k^{(1)} \in M$ such that $v_1 y_k^{(1)} = x_k^{(1)}$. Similarly, if $\{\bar{x}_k^{(1)}\}_{k\in N}$ is a set of generators of $\mathrm{Ker}(v_1|K)$, then there exist elements $\bar{y}_k^{(1)}$ such that $u_1 \bar{y}_k^{(1)} = \bar{x}_k^{(1)}$. Let

$$K_1 = K + \sum_{k\in N} R y_k^{(1)} + \sum_{k\in N} R \bar{y}_k^{(1)}.$$

We have a monomorphism $K(u_1, v_1) \subseteq K_1(u_1, v_1)$ such that

(i) if $x \in K$ and $u_1 x = 0$, then there exists an element $y \in K_1$ with $v_1 y = x$;

(ii) if $z \in K$ and $v_1 z = 0$, then there exists an element $z_1 \in K_1$ with $u_1 z_1 = z$.

Continuing this procedure we define countably generated R-submodules $K \subseteq K_1 \subseteq K_2 \subseteq \ldots$ of M such that the complex $\bar{K}(u_1, v_1)$ is exact, where

$$\bar{K} = \bigcup_{i\in N} K_i.$$

Hence the countably generated R-module $\bar{K}$ satisfies the required conditions.

LEMMA 4. *Let M and R be as in Lemma 3. Then every countably generated submodule K of M can be embedded into a countably generated (u_1, v_1)-exact R-submodule $\tilde{K}$ of M such that $u_1\tilde{K} = \tilde{K} \cap u_1 M$.*

Proof. Let K be a countably generated submodule of M. First we observe that $u_1 X \subseteq u_1 M \cap X \subseteq X$ for every submodule X of M. Using Lemma 3 we define by induction a family K_i $(i \in N)$ of countably generated submodules of M such that

$$K_i \subseteq \bar{K}_i \subseteq K_{i+1} \quad \text{for each } i \in N.$$

$\bar{K}_i$ are (u_1, v_1)-exact, $K_i \cap u_1 M = u_1 K_i$. We set

$$K_0 = K \quad \text{and} \quad K_i = \bar{K}_{i-1} + \sum_{j\in N} R m_j^{(i)} \text{ for } i > 0,$$

where $\{m_j^{(i)}, j \in N\}$ is a family of elements in M such that $y_j^{(i)} = u_1 m_j^{(i)}$ $(j \in N)$ form a set of generators of $\bar{K}_{i-1} \cap u_1 M$ (Lemma 2). Put

$$\tilde{K} = \bigcup_{i\in N} K_i = \bigcup_{i\in N} \bar{K}_i.$$

It is easy to observe that $\tilde{K}$ satisfies the condition of the lemma.

Proof of Theorem 2. Let M be a $(\boldsymbol{u}, \boldsymbol{v})$-exact R-module. In order to prove the theorem it is sufficient to show that every countably generated submodule of M can be embedded into a countably generated $(\boldsymbol{u}, \boldsymbol{v})$-exact submodule of M. Recall that $(\boldsymbol{u}, \boldsymbol{v})$ is the sequence of pairs $(u_1, v_1), \ldots, (u_n, v_n)$. We apply induction.

If $n = 1$, then the result follows from Lemma 3.

Now assume that $n > 1$ and put $M_1 = M/u_1M$. Let K be a countably generated submodule of M. By Lemma 4 there exists a countably generated (u_1, v_1)-exact submodule $\tilde{K}_1$ of M containing K such that $\tilde{K}_1 \cap u_1M = u_1\tilde{K}_1$. Let $p\colon M \to M_1$ denote the natural epimorphism. Since M_1 is $\big((u_2, v_2), \ldots, (u_n, v_n)\big)$-exact, by our assumption there exists a countably generated $\big((u_2, v_2), \ldots, (u_n, v_n)\big)$-exact submodule $M_1^{(1)}$ of M_1 such that $p_1(\tilde{K}_1) \subseteq M_1^{(1)}$. Let $\{x_n\}_{n\in N}$ be a set of elements in M such that the elements $p_1(x_n)$ $(n \in N)$ generate $M_1^{(1)}$. By Lemma 4 there exists a countably generated (u_1, v_1)-exact submodule $\tilde{K}_2$ of M such that $u_1\tilde{K}_2 = u_1M \cap \tilde{K}_2$. As previously, there exists a countably generated $\big((u_2, v_2), \ldots, (u_n, v_n)\big)$-exact submodule $M_1^{(2)}$ of M_1 such that $p_1(\tilde{K}_2) \subseteq M_1^{(2)}$. In such a way we define two chains of countably generated R-modules,

$$\tilde{K}_1 \subseteq \tilde{K}_2 \subseteq \tilde{K}_3 \subseteq \ldots \subseteq \tilde{K}_s \subseteq \ldots \subseteq M$$

and

$$M_1^{(1)} \subseteq M_1^{(2)} \subseteq \ldots \subseteq M_1^{(s)} \subseteq \ldots \subseteq M_1,$$

such that $M_1^{(s)}$ is $\big((u_2, v_2), \ldots, (u_n, v_n)\big)$-exact, $u_1\tilde{K}_s = u_1M \cap \tilde{K}_s$, $p_1(\tilde{K}_s) \subseteq M_1^{(s}$ and $\tilde{K}_s$ is (u_1, v_1)-exact for each k.

Put

$$M_0 = \bigcup_{s\in N} \tilde{K}_s.$$

First observe that M_0 is countably generated and contains K. Further, M_0 is (u_1, v_1)-exact and $u_1M \cap M_0 = u_1M_0$. Moreover, since

$$p_1(\tilde{K}_1) \subseteq M_1^{(1)} \subseteq p_1(\tilde{K}_2) \subseteq M_1^{(2)} \subseteq \ldots,$$

we have the following isomorphism:

$$M_0/u_1M_0 = M_0/u_1M \cap M_0 \simeq \bigcup_{s\in N} p_1(\tilde{K}_s) = \bigcup_{s\in N} M_1^{(s)}.$$

Hence M_0/u_1M_0 is $\big((u_2, v_2), \ldots, (u_n, v_n)\big)$-exact and, therefore, M_0 satisfies the required condition. Thus the theorem is proved.

COROLLARY. *Let (u, v) be a sequence of pairs $(u_1, v_1), \ldots, (u_n, v_n)$ in a commutative ring R. If R is (u, v)-exact Noetherian and every countably generated (u, v)-exact R-module is flat, then R is quasi-Frobenius.*

For the proof apply Theorem 2 and arguments in the proof of Theorem 2.3 in [2].

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Reçu par la Rédaction le 26. 2. 1977