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TOPOLOGICAL SEMIGROUPS AND KIRK'S QUESTION

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A *G-space*, in the terminology of Busemann [1], is a metric space M which is finitely compact (i.e., each bounded and closed subset of M is compact), metrically convex, and has a unique local prolongation.

In [8] Kirk posed the following question: If f is an isometry of a G-space M on itself and if some subsequence of $\{f^n(x)\}_{n=0}^{\infty}$, $x \in M$, is bounded, then is the sequence $\{f^n(x)\}_{n=0}^{\infty}$ bounded?

In [2] we have proved that even in a more general situation the answer to Kirk's question is positive (namely, such is the case of an arbitrary nonexpansive mapping f of a finitely compact metric space M into itself). In this note we shall use some facts on topological semigroups to give an alternative proof, which covers also the case of positively regular mappings on locally compact metric spaces. Moreover, we obtain results concerning positively regular mappings which may be considered as generalizations of [3].

We shall need here some basic definitions and facts concerning topological semigroups and positively regular mappings (see [3]–[5] for details).

A topological semigroup (resp., group) S is called *monothetic* ([6], § 9) if there exists an element $a \in S$, called a *generator* of S, such that the set $\{a^n\}_{n=1}^{\infty}$ (resp., $\{a^n\}_{n=-\infty}^{\infty}$) is dense in S. Clearly, each monothetic topological semigroup (resp., group) is commutative. It is also obvious that each topological group which is a monothetic topological semigroup is a monothetic topological group. The following proposition is implied by Theorem 10 (p. 40) in [7] and Theorem (9.1) in [6]:

PROPOSITION 1. *Let S be a locally compact monothetic topological semigroup and an abstract group. Then S is a compact monothetic topological group.*

Now, let f be a mapping of a metric space M (with a metric d) into itself. Then f is called *positively regular at a point* $x \in M$ ([4], cf. also [3]) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$ for each integer $n \geqslant 0$. The mapping f is called *positively regular on a set* $A \subset M$ if it is positively regular at each point of A. If f is positively regular on A and if the number δ does not depend on $x \in A$, then f is said to be *uniformly positively regular on* A. The mapping f is *nonexpansive* on A if $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in A$. Clearly, if $f(A) \subset A$ and if f is nonexpansive on A, then f is uniformly positively regular on A. On the other hand, let us prove the following remetrization lemma:

LEMMA 1. *Suppose f is positively regular on a set $A \subset M$ and $f(A) \subset A$. Then there exists a metric d_1 on A inducing the same topology and such that f is nonexpansive on A with respect to d_1.*

Proof. For all $x, y \in A$, define d_1 by

$$d_1(x, y) = \min\{\sup_{n \geqslant 0} d(f^n(x), f^n(y)); 1\}.$$

It is easily seen that d_1 is a metric on the set A and that f is nonexpansive on A with respect to d_1. Moreover, for all $x, y \in A$, $d_1(x, y) < 1$ implies $d(x, y) \leqslant d_1(x, y)$. Since f is positively regular on A, for each $x \in A$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies

$$d_1(x, y) \leqslant \sup_{n \geqslant 0} d(f^n(x), f^n(y)) \leqslant \varepsilon.$$

Consequently, d_1 and d are topologically equivalent on A and the proof is complete.

For $x \in M$, let

$$O(x) = \mathrm{cl}\{f^n(x)\}_{n=1}^{\infty} \quad \text{and} \quad K(x) = \bigcap_{n \geqslant 0} O(f^n(x)).$$

It follows from [3], Corollary 28, that if f is uniformly positively regular on $O(x)$ and if $O(x)$ is complete, then $O(x)$ with the well-defined multiplication

$$y \cdot z = \lim_{i \to \infty} f^{n_i + m_i}(x), \tag{1}$$

where $\{n_i\}$ and $\{m_i\}$ are sequences of positive integers such that

$$y = \lim_{i \to \infty} f^{n_i}(x) \quad \text{and} \quad z = \lim_{i \to \infty} f^{m_i}(x),$$

is a monothetic topological semigroup with $f(x)$ as a generator. Moreover, if $O(x)$ is compact, then, by Corollary 29 in [3], $K(x)$ with multiplication defined by (1) is a monothetic topological group and a minimal ideal in $O(x)$. This together with Lemma 1 yield

PROPOSITION 2. *Suppose f is positively regular on $O(x)$ and $O(x)$ is complete. Then $O(x)$ is a monothetic topological semigroup. Moreover, if $O(x)$ is compact, then $K(x)$ is a monothetic topological group and a minimal ideal in $O(x)$.*

Remark. It should be noted that even though $K(x)$ might be equal to $K(y)$ for some $x, y \in M$, $x \neq y$, their semigroup structures might be different (cf. the Remark to Proposition 3 below).

As a consequence of [5] we have the following

LEMMA 2. *Suppose f is nonexpansive on $O(x)$ and $y \in K(x)$. Then*

(i) *$O(y) = K(y)$ and $y \in K(y)$,*

(ii) *f maps $O(y)$ isometrically onto a dense subset of itself.*

Consequently, if $O(y)$ is complete, then f maps $O(y)$ isometrically onto itself.

Proof. (i) Clearly, $K(y) \subset O(y)$. By Proposition 1 in [5] we have $y \in K(y)$. Since $K(y)$ is invariant and closed, we obtain $O(y) \subset K(y)$.

(ii) By Theorem 1 in [5], the set $\{f^n(y)\}_{n=0}^{\infty}$ is mapped by f isometrically into itself, hence so is $O(y)$. Since $y \in O(y)$, $f(O(y))$ is dense in $O(y)$ and the assertion follows.

PROPOSITION 3. *Suppose f is positively regular on $O(x)$ and $O(x)$ is complete. If $y \in K(x)$, then*

(i) *$O(y) = K(y)$ and $y \in K(y)$,*

(ii) *$O(y)$ is a monothetic topological semigroup and an abstract group.*

Proof. (i) follows from Lemma 1 and from (i) of Lemma 2.

(ii) Since $O(x)$ is complete, so is $O(y)$. By Proposition 2, $O(y)$ (with multiplication defined by (1) at the point y!) is a monothetic topological semigroup. We show that $O(y)$ is an abstract group. Let

$$z = \lim_{i\to\infty} f^{n_i}(y),$$

where $\{n_i\}$ is a sequence of positive integers. Observe that

$$t \cdot z = \lim_{i\to\infty} f^{n_i}(t) \quad \text{for each } t \in O(y). \tag{2}$$

Indeed, let $t \in O(y)$ and let $\{m_j\}$ be a sequence of positive integers with $t = \lim_{j\to\infty} f^{m_j}(y)$. Since the multiplication in $O(y)$ is continuous, we have

$$t \cdot z = \lim_{i\to\infty} \big(t \cdot f^{n_i}(y)\big),$$

and since f is continuous for each integer i, we have

$$\begin{aligned} t \cdot f^{n_i}(y) &= \lim_{j\to\infty} \big(f^{m_j}(y) \cdot f^{n_i}(y)\big) = \lim_{j\to\infty} f^{n_i}\big(f^{m_j}(y)\big) \\ &= f^{n_i}\big(\lim_{j\to\infty} f^{m_j}(y)\big) = f^{n_i}(t), \end{aligned}$$

which implies (2).

Now, setting $t = y$ in (2), we obtain $y \cdot z = z$. Thus y is the identity of $O(y)$. To complete the proof, it remains to show that there exists an element

$t \in O(y)$ such that $t \cdot z = y$. By Lemma 1, we may assume that f is nonexpansive on $O(x)$ and, by (ii) of Lemma 2, that f maps $O(y)$ isometrically onto itself. Thus there exists a sequence $\{t_i\}$ of points of $O(y)$ such that $f^{n_i}(t_i) = y$ for each integer i. Moreover, for all integers i, j, we have

$$d(t_i, t_j) = d\big(f^{n_i+n_j}(t_i), f^{n_i+n_j}(t_j)\big) = d\big(f^{n_j}(y), f^{n_i}(y)\big).$$

Hence $\{t_i\}$ is a Cauchy sequence. Since $O(y)$ is complete, there exists $t = \lim_{i\to\infty} t_i$. Thus,

$$t \cdot z = \lim_{i\to\infty} \big(t_i \cdot f^{n_i}(y)\big)$$

and, by (2), $t_i \cdot f^{n_i}(y) = f^{n_i}(t_i) = y$ for each i. Therefore, $t \cdot z = y$, which completes the proof.

Remark. Under the assumptions of Proposition 3, $K(x) = O(y) = K(y)$. Indeed, it suffices to show that $K(x) \subset K(y)$. Let $\{n_i\}$ be a sequence of positive integers with $y = \lim_{i\to\infty} f^{n_i}(x)$. If $z \in K(x)$, then there exists a strictly increasing sequence $\{m_i\}$ of positive integers such that $z = \lim_{i\to\infty} f^{m_i}(x)$. Replacing $\{m_i\}$ by a suitable subsequence, we may assume that $m_i - n_i \geq 1$ for each integer i. By Lemma 1, we may also assume that f is nonexpansive on $O(x)$. Consequently, for each integer i we have

$$\begin{aligned} d\big(z, f^{m_i-n_i}(y)\big) &\leq d\big(z, f^{m_i}(x)\big) + d\big(f^{m_i}(x), f^{m_i-n_i}(y)\big) \\ &\leq d\big(z, f^{m_i}(x)\big) + d\big(f^{n_i}(x), y\big). \end{aligned}$$

This shows that $z = \lim_{i\to\infty} f^{m_i-n_i}(y)$, whence $z \in K(y)$. However, this fact is not needed in this note.

It is easily seen that if $x \in O(x)$, then $x \in K(x)$. Thus Proposition 3 yields

COROLLARY 1. *Under the assumptions of Proposition* 3, *if* $x \in O(x)$, *then* $O(x) = K(x)$ *is a monothetic topological semigroup and an abstract group.*

We may now prove the following

THEOREM 1. *Suppose* f *is positively regular on* $O(x)$ *and* $O(x)$ *is locally compact and complete. If* $K(x) \neq \emptyset$ (*i.e., if the sequence* $\{f^n(x)\}_{n=0}^{\infty}$ *has a convergent subsequence*), *then*

(i) $O(x)$ *is a compact monothetic topological semigroup,*

(ii) $K(x)$ *is a compact monothetic topological group and a minimal ideal in* $O(x)$.

Proof. In view of Proposition 2 it suffices to prove that $O(x)$ is compact. Let $y \in K(x)$. By Proposition 3, $O(y) = K(y)$, $y \in K(y)$, and $O(y)$ is a monothetic topological semigroup and an abstract group. Since $O(x)$ is locally compact, so is $O(y)$. Thus, it follows from Proposition 1 that $O(y)$ is compact. Therefore, there exists a number $r > 0$ such that the set

$$U = \{z \in O(x)\colon d(O(y), z) \leqslant r\}$$

is a compact neighbourhood of $O(y)$ in $O(x)$. Moreover, by Lemma 1, we may assume that f is nonexpansive on $O(x)$, and hence that $f(U) \subset U$. Since $y \in K(x) \cap U$, there exists a positive integer n such that $f^n(x) \in U$. Thus $O(f^n(x)) \subset U$. Hence $O(f^n(x))$ is compact, and therefore so is

$$O(x) = \{f(x), \ldots, f^n(x)\} \cup O(f^n(x)).$$

This completes the proof.

As an immediate consequence of (i) of Theorem 1, we have

COROLLARY 2. *Under the assumptions of Theorem* 1, *the sequence* $\{f^n(x)\}_{n=0}^{\infty}$ *is bounded.*

Clearly, each bounded sequence of points of a finitely compact set has a convergent subsequence. Thus, Theorem 1 implies (cf. the question of Kirk [8])

COROLLARY 3. *Suppose f is positively regular on $O(x)$ and $O(x)$ is finitely compact. If the sequence $\{f^n(x)\}_{n=0}^{\infty}$ has a bounded subsequence, then $O(x)$ is compact, and hence bounded.*

In the case of connected finitely compact metric spaces, we get (note that G-spaces are connected and finitely compact)

THEOREM 2. *Let f be a positively regular mapping of a connected finitely compact metric space M into itself. Suppose there exists a point $y \in M$ such that the sequence $\{f^n(y)\}_{n=0}^{\infty}$ has a bounded subsequence. Then, for each $x \in M$, $O(x)$ is compact, and hence the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is bounded.*

Outline of the proof. By Corollary 3, $\{f^n(y)\}_{n=0}^{\infty}$ is bounded. Observe that if f is nonexpansive, then for each $x \in M$

$$\operatorname{diam}_d(\{f^n(x)\}_{n=0}^{\infty}) \leqslant \operatorname{diam}_d(\{f^n(y)\}_{n=0}^{\infty}) + 2d(x, y),$$

and so $\{f^n(x)\}_{n=0}^{\infty}$ is bounded. Thus the proof will be completed if we construct a new metric d_0 on M such that

(a) f is nonexpansive on M with respect to d_0,

(b) for each $x \in M$, $\operatorname{diam}_{d_0}(\{f^n(x)\}_{n=0}^{\infty}) = \operatorname{diam}_d(\{f^n(x)\}_{n=0}^{\infty})$.

Define d_0 by

$$d_0(x, y) = \sup_{n \geqslant 0} d(f^n(x), f^n(y)) \quad \text{for } x, y \in M.$$

It follows from the proof of Lemma 1 that each point of M has a neighbourhood on which d_0 is finite. Since M is connected and d_0 satisfies the triangle inequality, d_0 is finite on the whole space M. Thus d_0 is a metric on M and we infer that d_0 satisfies both the conditions (a) and (b).

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