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A NOTE ON A GENERALIZATION OF THE STURM-PICONE THEOREM

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1. Introduction. The classical form of the Sturm-Picone theorem compares equations

$$(ax')' + cx = 0 \quad (t \geqslant 0), \tag{1}$$

$$(Au')' + Cu = 0 \quad (t \geqslant 0), \tag{2}$$

where a, A, and c, C are functions of classes C^1 and C^0, respectively. Let the inequalities $a \leqslant A$ and $c \geqslant C$ be true on $[t_1, t_2] \subset (0, \infty)$. Then if any solution of (2) has two (or more) zeros on $[t_1, t_2]$, then every solution of (1) also has at least one zero on $[t_1, t_2]$. (By a solution we mean a classical solution.)

Generalizations exist with replacing these inequalities by certain averages of the functions A, a, C, c.

In this note we pursue a related topic. Can $A \geqslant a$ and $C \leqslant c$ be replaced by $A\varphi \geqslant a$ and $C\varphi \leqslant c$ for a suitable class of functions $\varphi \in C^1[t_1, t_2]$? In particular, for what class of functions φ ($\varphi \in C^2$) is the following theorem true?

If (2) *has two zeros on* $[t_1, t_2]$, *then* (1) *will have at least one zero on* $[t_1, t_2]$ *whenever* $A = a\varphi$ *and* $C = c\varphi$.

This note does not offer a characterization of such a class of functions, and only a partial answer is given.

Theorems 1-3 in Section 2 are of local character, specifically determining existence or non-existence of zeros on a fixed interval $[t_1, t_2]$. Global oscillation — non-oscillation theorems can be deduced as corollaries to Theorems 1-3. We comment that Kummer's transformation [3] could be used in the arguments given in this paper, offering an alternative method of proof.

2. Main theorems and an example of application. The oscillatory behavior of solutions of the equation

$$(a(t)x')' + c(t)x = 0, \quad t \geqslant t_0, \tag{3}$$

is compared with that of solutions of

$$(a(t)\varphi(t)u')' + c(t)\varphi(t)u = 0, \quad t \geqslant t_0, \ \varphi \in C^1[t_0, \infty]. \tag{4}$$

First we prove a Sturm-type theorem concerning the existence of zeros on an interval $[t_1, t_2]$.

THEOREM 1. *Suppose that there exists a solution $\hat{u}(t)$ of equation* (4) *which has no zeros on $[t_1, t_2]$. If $a(t)\varphi(t) > 0$ on $[t_1, t_2]$ and $(a\varphi')' \leqslant 0$ but $(a\varphi')' \neq 0$ on $[t_1, t_2]$, then any solution $\hat{x}(t)$ of* (3) *can have at most one zero on $[t_1, t_2]$* ([1]).

Proof. Suppose that there exists a solution $\hat{x}(t)$ of (3) which has zeros at $t = \tau_1$ and $t = \tau_2$ on the interval $[t_1, t_2]$. Assume without loss of generality that $\hat{x}(t) > 0$ on (τ_1, τ_2). Consider the integral

$$\int_{\tau_1}^{\tau_2} (a(t)\varphi(t)\hat{x}\hat{x}')'\,dt,$$

which is clearly equal to zero. However,

$$\int_{\tau_1}^{\tau_2} (a(t)\varphi(t)\hat{x}(t)\hat{x}'(t))'\,dt$$
$$= \int_{\tau_1}^{\tau_2} \left\{\frac{1}{2}a(t)\varphi'(t)\frac{d}{dt}(\hat{x}^2) + a(t)\varphi(t)(\hat{x}')^2 + \varphi(t)\hat{x}(t)(a\hat{x}')'\right\}dt$$
$$= -\frac{1}{2}\int_{\tau_1}^{\tau_2} [(a\varphi')'\hat{x}^2]\,dt + \int_{\tau_1}^{\tau_2} [(a\varphi)(\hat{x}')^2 - (c\varphi)\hat{x}^2]\,dt.$$

Since $(a\varphi')' \leqslant 0$ on $[t_1, t_2]$ and $(a\varphi')' \not\equiv 0$, we have

$$\int_{\tau_1}^{\tau_2} [(a\varphi)(\hat{x}')^2 - (c\varphi)\hat{x}^2]\,dt < 0. \tag{5}$$

Hence there exists a function $\hat{x}(t) \in C^1[t_1, t_2]$ vanishing at $t = t_1$ and $t = t_2$ and satisfying inequality (5). An application of Leighton's variational theorem (see [4]) shows that every solution $u(t)$ of (4) has a zero on $[t_1, t_2]$. This is a contradiction which proves the theorem.

([1]) This statement is the best possible.

An obvious corollary states that if (4) is non-oscillatory, then so is (3).

THEOREM 2. *Subject to the same hypothesis, any solution of* (4) *has a zero on any interval containing two zeros of* (3).

Proof. Suppose to the contrary that a solution of (4) has no zeros on such an interval. Now an application of Theorem 1 results in a contradiction.

Additional corollaries can be obtained by combining this result with the main theorem of [2].

THEOREM 3. *Suppose that a solution $\hat{x}(t)$ of the equation*

$$(a(t)x')' + c(t)x = g(t)f(x), \qquad t \geqslant t_0, \tag{6}$$

has zeros at $t = t_1 > t_0$ and $t = t_2 > t_1 > t_0$. We assume that $\xi f(\xi) > 0$ if $\xi \neq 0$, $f \in C(-\infty, +\infty)$, $g \in L_1[t_0, \infty)$ locally. If there exists a function $\varphi(t) \in C^1[t_1, t_2]$ such that $a(t)\varphi(t) > 0$, $(a\varphi')' \leqslant 0$, $g\varphi \geqslant 0$, $g\varphi \in L_1$ (locally), with either $(a\varphi')'$ or $g\varphi$ not vanishing identically on $[t_1, t_2]$, and if $A \leqslant a\varphi$ and $C \geqslant c\varphi$ on $[t_1, t_2]$, then any solution of (2) *will have a zero on $[t_1, t_2]$.*

Proof. Following the argument of Theorem 1 we have

$$0 = \int_{t_1}^{t_2} (a\varphi\hat{x}\hat{x}')'dt = -\frac{1}{2}\int_{t_1}^{t_2} [(a\varphi')'\hat{x}^2 - g\varphi\hat{x}f(\hat{x})]dt + \int_{t_1}^{t_2} [a\varphi(\hat{x}')^2 - c\varphi\hat{x}^2]dt.$$

Hence

$$\int_{t_1}^{t_2} [(a\varphi)(\hat{x}')^2 - (c\varphi)\hat{x}^2]dt < 0.$$

The remainder of the argument is as before.

I comment that this idea of proof occurred to me after reading the proof of Theorem 1 of [1].

COROLLARY 1. *If (subject to the same hypothesis) any solution of* (2) *has no zeros on the interval $[t_1, t_2]$, then any solution of* (6) *will have at most one zero on $[t_1, t_2]$.*

Comment. If the condition "$xf(x) > 0$ if $x \neq 0$" is replaced by "$xf(x) < 0$ if $x \neq 0$", then Theorem 3 remains true whenever $g\varphi \geqslant 0$ is also replaced by the condition $g\varphi \leqslant 0$, the remaining hypothesis being unchanged.

Examples. (a) Any solution of the equation

$$(\sin t \cdot u')' + K^2 \sin t \cdot u = 0, \qquad K > 2,$$

has a zero on the interval $[0, \pi]$.

To prove this statement we compare it with $x'' + K^2x = 0$, which has solutions containing two zeros on $[0+\varepsilon, \pi-\varepsilon]$ for a sufficiently small $\varepsilon > 0$. Here $a = 1$, $c = K^2$, and $\varphi = \sin t$. The expression $(a\varphi')' = -\sin t$ is negative on $[\varepsilon, \pi-\varepsilon]$ while $a\varphi = \sin t$ is positive. Now, Theorem 2 states that every solution of this equation has a zero on $[\varepsilon, \pi-\varepsilon]$, hence on $[0, \pi]$.

(b) Consider the equations

$$u'' + u = 0$$

and

$$x'' + \frac{1}{2}(x - x^3 \sin t) = 0. \tag{7}$$

Equation (7) can be rewritten as

$$(2x')' + x = x^3 \sin t,$$

i.e. we identify $a \equiv 2$, $c \equiv 1$, $g \equiv \sin t$, and $f(x) \equiv x^3$. We claim that any solution of (7) can have at most one zero on the interval $[\pi/3, \pi/2]$. We check the hypothesis of Theorem 3 after choosing $\varphi(t) = \sin t$.

We have

$$a\varphi = 2\sin t \geqslant 1 \equiv A, \quad c\varphi = \sin t \leqslant 1 \equiv C, \quad (a\varphi')' = -2\sin t < 0,$$
$$g\varphi = \sin^2 t > 0, \quad xf(x) = x^4 > 0 \text{ if } x \neq 0.$$

Hence Corollary 1 is applicable, since there exists a solution of $u'' + u = 0$ having no zeros on $[\pi/3, \pi/2]$.

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