COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 1

ON ATOMIC MAPPINGS

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A continuum means in this paper a compact connected Hausdorff space. A continuous map $f: X \to Y$ is said to be *monotone* provided the preimage under f of any point of Y is connected. It is known (e. g., [6], p. 123) that f is monotone iff the preimage under f of any subcontinuum of Y is connected. In Cook's paper [3] a continuous map $f: X \xrightarrow{\text{onto}} Y$ is called *preatomic* iff, for each subcontinuum K of X such that $f(K)$ is non-degenerate (i. e., containing more than one point), there is $f^{-1}(f(K)) = K$; a continuous map $f: X \xrightarrow{\text{onto}} Y$ is said to be *atomic* provided it is preatomic and monotone. The notion of atomic continuous (and open) maps was introduced by Anderson [1] and was applied by Anderson and Choquet [2] and by Cook [3] to constructions of some singular continua. We shall prove that the two notions, of atomic and of preatomic continuous maps of continua, are equivalent and we shall discuss the problem of characterization of those continua X for which each atomic map $f: X \xrightarrow{\text{onto}} Y$, where Y is Hausdorff, is a homeomorphism. For instance, we shall show that all arcwise connected continua, as well as all aposyndetic continua, have this property. Another theorem gives a characterization of hereditarily indecomposable continua in terms of atomic maps.

1. Preliminaries. Throughout this paper all maps are assumed to be continuous and all spaces are assumed to be Hausdorff.

THEOREM 1. *Let X be a continuum. If $f: X \xrightarrow{\text{onto}} Y$ is preatomic, then it is atomic.*

Proof. It suffices to prove the monotoneity of f. Let $y \in Y$. Let M and N denote closed subsets of $f^{-1}(y)$ such that $f^{-1}(y) = M \cup N$, $M \cap N = 0$ and $M \neq 0$. Since X is normal, there exist subsets U and V of X such that $M \subset U, N \subset V$ and $U \cap V = 0$. Let $x \in M$ and let K denote the component of $\overline{U}$ containing x. From Janiszewski's Lemma (cf. [6], p. 112) it follows that $K \cap \mathrm{Fr}\, \overline{U} \neq 0$. Let $z \in K \cap \mathrm{Fr}(\overline{U})$. Since $\overline{U} \cap V = 0$ and $\mathrm{Fr}\,(\overline{U}) = \overline{U} - U$, we infer that $z \notin f^{-1}(y)$. Hence $f(K)$ is non-degenerate and, therefore, $f^{-1}(f(K)) = K$. Since $y = f(x) \in f(K)$, we have $M \cup N = f^{-1}(y) \subset K \subset \overline{U}$. Thus $N \subset K$ and $N \cap K = 0$. Hence $N = 0$, and, therefore, $f^{-1}(y)$ is connected.

A subset C of X is said to be a *semi-continuum* provided for each two points a and b of C there is a subcontinuum K of X containing a and b and such that $K \subset C$.

THEOREM 2. *Let f be an atomic map from X onto Y. If C is a semi-continuum of X such that $f(C)$ is non-degenerate, then $f^{-1}(f(C)) = C$.*

Proof. Let $y \in f(C)$. Since $f(C)$ is non-degenerate, there exist two points x and z of C such that $y = f(x) \neq f(z)$. Let K be a subcontinuum of X containing x and z and such that $K \subset C$. Then $f(K)$ is a non-degenerate subcontinuum of Y and (since f is atomic) $f^{-1}(y) \subset f^{-1}(f(K)) = K \subset C$. Hence $f^{-1}(f(C)) \subset C$ and, in virtue of $C \subset f^{-1}(f(C))$, we get $f^{-1}(f(C)) = C$.

THEOREM 3. *A map $f\colon X \xrightarrow{\text{onto}} Y$ is atomic iff for each $y \in Y$ and each semi-continuum C of X, provided $C \cap f^{-1}(y) \neq 0$ and $C \cap (X - f^{-1}(y)) \neq 0$, then $f^{-1}(y) \subset C$.*

Proof. 1. Let the map f be atomic and $y \in Y$. If C is a semi-continuum of X such that $C \cap f^{-1}(y) \neq 0$ and $C \cap (X - f^{-1}(y)) \neq 0$, then $f(C)$ is non-degenerate and, therefore, by Theorem 2, $f^{-1}(f(C)) = C$. Since $y \in f(C)$, we get $f^{-1}(y) \subset C$.

2. Let K be a subcontinuum of X such that $f(K)$ is non-degenerate. Let $x \in f^{-1}(f(K))$. Since $f(x) \in f(K)$, we infer that $f^{-1}(f(x)) \cap K \neq 0$ and $K \cap (X - f^{-1}(f(x))) \neq 0$. Being a continuum, K is a semi-continuum, hence $x \in f^{-1}(f(x)) \subset K$; then from $K \subset f^{-1}(f(K))$ we get $f^{-1}(f(K)) = K$. Thus f is atomic.

COROLLARY 1. *A map $f\colon X \xrightarrow{\text{onto}} Y$ is atomic iff, for each semi-continuum C of X, there is $C \subset f^{-1}(y)$ for a $y \in Y$ or $C = \bigcup\{f^{-1}(y)\colon y \in M\}$ for some subset M of Y.*

Proof. 1. Let C be a semi-continuum of X and $M = \{y \in Y\colon {}^{-1}(y) \cap \cap C \neq 0\}$. If $M = \{y\}$, then $C \subset f^{-1}(y)$. If M contains at least two points, then from Theorem 3 it follows that we have $f^{-1}(y) \subset C$ for each $y \in M$. This implies that $\bigcup \{f^{-1}(y)\colon y \in M\} \subset C$. Thus from $C \subset \bigcup \{f^{-1}(y)\colon y \in M\}$ we get $C = \bigcup \{f^{-1}(y)\colon y \in M\}$.

2. Let C be a semi-continuum of X such that $C \cap f^{-1}(y) \neq 0$ and $C \cap (X - f^{-1}(y)) \neq 0$. Then, by hypothesis, we infer that $f^{-1}(y) \subset C$. Hence, by Theorem 3, we conclude that f is atomic.

COROLLARY 2. *There exists an atomic map f of continuum X onto a non-degenerate Y iff there exists an upper semi-continuous monotone decomposition $\mathscr{D}$ of X such that*

(i) *for each semi-continuum C of X, we have $C \subset K$ for a K from $\mathscr{D}$ or $C = \bigcup \mathscr{D}'$, where $\mathscr{D}' \subset \mathscr{D}$.*

Proof. If the map f is atomic, then, by Corollary 1, the decomposition $\{f^{-1}(y): y \in Y\}$ satisfies (i) and it is upper semi-continuous (this follows easily from the fact that f is closed). If there exists an upper semi-continuous decomposition $\mathscr{D}$ which satisfies (i), then the corresponding quotient map is continuous and, by Corollary 1, atomic.

COROLLARY 3. *Let f be an atomic map from a continuum X onto a non-degenerate Y. Let $y \in Y$ and let K be a component of $X - f^{-1}(y)$. Then $f^{-1}(y) \subset \bar{K}$ and $\mathrm{Int}_{\bar{K}} f^{-1}(y) = 0$ (hence $\mathrm{Int}_X f^{-1}(y) = 0$).*

Proof. From the hypothesis and Janiszewski's Lemma (cf. [6], p. 112) it follows that $\bar{K} \cap \mathrm{Fr}(X - f^{-1}(y)) \neq 0$. Since $X - f^{-1}(y)$ is an open subset of X, we get $\mathrm{Fr}(X - f^{-1}(y)) \subset f^{-1}(y)$. Hence $f^{-1}(y) \cap \bar{K} \neq 0$ and $(X - f^{-1}(y)) \cap \bar{K} \neq 0$. From Theorem 3 we infer that $f^{-1}(y) \subset \bar{K}$. Since $f^{-1}(y) \cap K = 0$, we have $f^{-1}(y) \subset \mathrm{Fr}(\bar{K})$ in $\bar{K}$ and $\mathrm{Int}_{\bar{K}} f^{-1}(y) = 0$.

COROLLARY 4. *Let f be an atomic map from a continuum X onto a non-degenerate Y. Let $B \subset f^{-1}(y)$ and $B \neq f^{-1}(y)$. Then $X - B$ is connected.*

Proof. Let K be a component of $X - f^{-1}(y)$. Since $K \subset \bar{K} \cap (X - B) \subset \bar{K}$, we infer that $\bar{K} \cap (X - B)$ is connected. From Corollary 3 it follows that $f^{-1}(y) - B \subset \bar{K}$. This implies that the set $\bigcup \{\bar{K} \cap (X - B): K$ is a component of $X - f^{-1}(y)\}$ is equal to $X - B$ and that it is connected. Thus $X - B$ is connected.

The fact that if f is an atomic map from X onto Y and K is a subcontinuum of X such that $f(K)$ is non-degenerate, then $f \mid K: K \xrightarrow{\text{onto}} f(K)$ is an atomic map, implies the following two corollaries:

COROLLARY 5. *Let f be an atomic map from a continuum X onto a non-degenerate Y. Let $y \in Y$ and S be a subcontinuum of X such that $f(S)$ is non-degenerate and $y \in f(S)$. If K is a component of $S - f^{-1}(y)$, then $f^{-1}(y) \subset \bar{K}$ and $\mathrm{Int}_{\bar{K}} f^{-1}(y) = 0$.*

COROLLARY 6. *Let the hypotheses of Corollary 4 hold. If K is a subcontinuum of X such that $f(K)$ is non-degenerate, then $K - B$ is connected.*

2. A characterization of hereditarily indecomposable continua. We shall use the following result: a continuum X is hereditarily indecomposable iff for each two subcontinua K_1 and K_2 of X such that $K_1 \cap K_2 \neq 0$ and $K_1 \not\subset K_2$ there is $K_2 \subset K_1$.

THEOREM 4. *A continuum X is hereditarily indecomposable iff each continuous and monotone map from X is atomic.*

Proof. 1. Let X be a hereditarily indecomposable continuum and let a map f from X be continuous and monotone. Let K be a subcontinuum of X such that $f(K)$ is non-degenerate. Since $M = \{y: f^{-1}(y) \cap K \neq 0\}$ is non-degenerate, we have $K \not\subset f^{-1}(y)$ for every $y \in M$, and since X is hereditarily indecomposable, there is $f^{-1}(y) \subset K$. This implies that

$$f^{-1}(f(K)) = \bigcup\{f^{-1}(y)\colon y \in M\} \subset K \quad \text{and} \quad K = f^{-1}(f(K)).$$

Hence f is atomic.

2. Let K_1 and K_2 be subcontinua of X such that $K_1 \cap K_2 \neq 0$ and $K_2 \not\subset K_1$. Let $\mathscr{D}$ be a decomposition of X consisting of K_1 and single points of $X - K_1$. The quotient map f from X onto $\mathscr{D}$ is continuous and monotone. This implies that f is atomic. Since $f(K_2)$ is non-degenerate and $f(K_1) \subset f(K_2)$, we get

$$K_1 = f^{-1}(f(K_1)) \subset f^{-1}(f(K_2)) = K_2.$$

Hence $K_1 \subset K_2$. Therefore, according to the result quoted above, the continuum X is hereditarily indecomposable.

3. Some cases where atomic maps are homeomorphisms. If f is a homeomorphism, then clearly f is atomic. There are simple examples of atomic maps which are not homeomorphisms. For example, let X be "the $\sin x^{-1}$ curve" and let f be a projection from X onto x-axis. We see that f is atomic.

A continuum X is *aposyndetic at x with respect to y*, $y \neq x$ (a notion of Jones, cf. [4] and [5]), provided there exists a subcontinuum of X containing x in its interior and not containing y. A continuum X is *aposyndetic* iff for each $x \in X$ it is aposyndetic at x with respect to every point of X.

THEOREM 5. *If f is an atomic map from an aposyndetic continuum X onto a non-degenerate Y, then it is a homeomorphism.*

Proof. Let $y \in Y$, $x \in f^{-1}(y)$ and $z \in X$ be such that $z \neq x$. There is a subcontinuum K of X containing x in its interior and not containing z. It follows from Corollary 3 that $f(K)$ is non-degenerate. Since f is atomic, we infer that $f^{-1}(f(K)) = K$. Hence $z \notin f^{-1}(y)$. Since z is an arbitrary point different from x, we get $f^{-1}(y) = \{x\}$, and, therefore, f is a homeomorphism.

Since locally connected continua are obviously aposyndetic, we get

COROLLARY 7. *If a map from a locally connected continuum X onto a non-degenerate Y is atomic, then it is a homeomorphism.*

A subset B of continuum X is said to be *arcwise connected* in X iff for each two distinct points a, b of B there exists a continuum $K \subset B$ and a homeomorphism $h\colon [0,1] \xrightarrow{\text{onto}} K$ such that $a, b \in K$ and $h(0) = 0, h(1) = b$.

THEOREM 6. *Let $\mathscr{P}$ be a covering of X by arcwise connected subsets of X. If f is an atomic map of X onto non-degenerate continuum Y, then for each point $y \in Y$ the set $f^{-1}(y)$ is either degenerate or $f^{-1}(y) = \bigcup \mathscr{P}'$, where $\mathscr{P}'$ is a subfamily of $\mathscr{P}$.*

Proof. Let y be a point of Y and let $\mathscr{M}$ be a subfamily of $\mathscr{P}$ consisting of those elements which meet $f^{-1}(y)$. It is sufficient to prove that $\bigcup\mathscr{M} \subset f^{-1}(y)$ if $f^{-1}(y)$ is non-degenerate. Let $B\in\mathscr{M}$. Suppose that $B-f^{-1}(y) \neq 0$. Then there exist two distinct points, x in $f^{-1}(y)\cap B$ and z in $B-f^{-1}(y)$, a continuum K, and a homeomorphism $h\colon [0,1]\xrightarrow{\text{onto}} K$ such that $h(0)=z$, $h(1)=x$. Let $t\in[0,1]$ be such that $h(t)\in f^{-1}(y)$ and $t'<t$ implies $h(t')\notin f^{-1}(y)$. Let $b=h(t)$. Then $K'=h([0,t])$ is a continuum such that $K'\cap f^{-1}(y)\neq 0$ and $K'\cap\big(X-f^{-1}(y)\big)\neq 0$, whence $f^{-1}(y)\subset K'$. Since $f^{-1}(y)\cap K'=\{b\}$, we have $f^{-1}(y)=\{b\}$ so that $f^{-1}(y)$ is degenerate.

COROLLARY 8. *If f is an atomic map of a continuum X onto Y and B is an arcwise connected subset of X, then either $f\mid B\colon B\xrightarrow{\text{onto}} f(B)$ is one-to-one map or $f(B)$ is degenerate.*

Proof. If $f\mid B$ is not one-to-one, then there exists a point y of $f(B)$ such that $(f\mid B)^{-1}(y)$ is non-degenerate and from Theorem 6 we get $B\subset f^{-1}(y)$.

COROLLARY 9. *If f is an atomic map of an arcwise connected continuum X onto a non-degenerate Y, then f is a homeomorphism.*

Proof. Since X is arcwise connected and $f(X)$ is non-degenerate, we infer from Corollary 8 that $f^{-1}(y)$ is degenerate.

Example 1. There exists a continuum X which is not arcwise connected and such that

(*) if f is an atomic map of X onto a non-degenerate Y, then f is a homeomorphism.

Namely, let X be the subset of the plane E^2 to which (x,y) belongs iff either $y=\sin x^{-1}$ for $x\in(0,1]$ or $x=0$ and $y\in[-1,2]$, with the topology induced from the plane E^2. Clearly, X is not arcwise connected. Property (*) follows from Theorem 6.

Example 2. It follows from Theorem 6 that each atomic map of Brouwer's indecomposable continuum (cf. [6], p. 143) onto a non-degenerate continuum Y is a homeomorphism.

4. The irreducibility of atomic maps. Let us begin with

THEOREM 7. *If f is an atomic map of a continuum X onto a non-degenerate Y, then for each proper subcontinuum K of X there is $f(K)\neq Y$; in the other words, f is irreducible with respect to subcontinua.*

Proof. Let K be a subcontinuum of X such that $f(K)=Y$. Since Y is non-degenerate, $K=f^{-1}\big(f(K)\big)=f^{-1}(Y)=X$.

There exist easy examples of maps which are irreducible in the usual sense (i. e., with respect to closed subsets) but which are not atomic. For example, let X be the subset of the plane E^2 to which (x,y) belongs iff either $y=\sin x^{-1}$ for $x\in(0,1]$ or $x=0$ and $y\in[-1,1]$ or $y=0$ and

$x \in [-1, 0)$, with the topology induced from the plane E^2. Let $Y = [-1, 1]$ and let $f\colon X \xrightarrow{\text{onto}} Y$ be the projection onto x-axis. By Theorem 6, f is not atomic.

On the other hand, we shall show that there exist atomic maps which are not irreducible.

Example 3. Let X be a hereditarily indecomposable continuum and $W \neq 0$ be an open subset of X such that $\overline{W} \neq X$. Let F be a decomposition of X consisting of the components of $\overline{W}$ and single points of $X - \overline{W}$. Then F is upper semi-continuous and the quotient map is continuous and monotone; hence we infer from Theorem 4 that f is atomic. But from Janiszewski's Lemma (cf. [6], p. 112), for each component K of $\overline{W}$, there is $K \cap (\overline{W} - W) = K \cap \mathrm{Fr}(\overline{W}) \neq 0$; hence $f(X - W) = Y$. Since $X - W$ is closed, f is not irreducible.

THEOREM 8. *If f is a map of an indecomposable continuum X onto a non-degenerate Y, which is monotone and irreducible with respect to continua, and if K_x is a composant of a point x of X, then $f(K_x)$ is a composant of $f(x)$ in Y.*

Proof. Let $K_x \subset X$ be a composant of x in X. Let y be a point of $f(K_x)$. Then there exists a point z of K_x such that $f(z) = y$. Hence, there exists a continuum K_{xz} such that $K_{xz} \subset K_x$, $x \in K_{xz}$, $z \in K_{xz}$, $K_{xz} \neq X$. Since $y \in f(K_{xz})$, $f(x) \in f(K_{xz})$ and $f(K_{xz}) \neq Y$ (f is irreducible with respect to continua), we have $y \in K_{f(x)}$, hence $f(K_x) \subset K_{f(x)}$. We shall show that if K_{x_1} and K_{x_2} are distinct composants, then $K_{f(x_1)}$ and $K_{f(x_2)}$ are also two distinct composants. Suppose $K_{f(x_1)} = K_{f(x_2)}$. Then there exists a subcontinuum K of Y such that $f(x_1) \in K$ and $f(x_2) \in K$ and $K \neq Y$. Then $x_1 \in f^{-1}(K)$, $x_2 \in f^{-1}(K)$ and $f^{-1}(K) \neq X$, where $f^{-1}(K)$ is a continuum (f is monotone). Hence $K_{x_1} = K_{x_2}$. Since f is a map onto Y, we have $f(K_x) = K_{f(x)}$.

COROLLARY 10. *If f is a monotone map of a hereditarily indecomposable continuum X onto a non-degenerate Y and if K_x is a composant of x in X, then $f(K_x)$ is a composant of $f(x)$ in Y.*

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Reçu par la Rédaction le 28. 5. 1971