COLLOQUIUM MATHEMATICUM

VOL. L 1985 FASC.1

ON CONVOLUTION PRODUCTS OF RADIAL MEASURES ON THE HEISENBERG GROUP

BY

KRZYSZTOF STEMPAK (WROCŁAW)

For $n = 1, 2, 3, \ldots$ we consider the $2n+1$ dimensional Heisenberg group $\boldsymbol{H}^n = \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}$ with the multiplication law

$$(x, y, t)\cdot(x', y', t') = \big(x+x', y+y', t+t'+\sum_{j=1}^{n} (x_j y'_j - x'_j y_j)\big).$$

The center of $\boldsymbol{H}^n$ is $\boldsymbol{Z}^n = \{(0, 0, t)\colon t \in \boldsymbol{R}\}$. The Haar measure of $\boldsymbol{H}^n$ is the ordinary Lebesgue measure. For $g \in SO(2n, \boldsymbol{R})$ we define the "rotation"

$$\bar{g}\colon \boldsymbol{H}^n \to \boldsymbol{H}^n, \quad \bar{g}(z, t) = \big(g(z), t\big), \quad (z, t) \in \boldsymbol{H}^n.$$

By a *radial measure* we mean a finite measure $\mu \in M(\boldsymbol{H}^n)$ such that for every $g \in SO(2n, \boldsymbol{R})$ $\bar{g}\cdot\mu = \mu$ where $g\cdot\mu(S) = \mu\big(g^{-1}(S)\big)$ for Borel sets $S \subseteq \boldsymbol{H}^n$.

The aim of this note is to prove that under some mild natural condition on a radial measure on $\boldsymbol{H}^n$ its third when $n = 1$ or second when $n \geqslant 2$ convolution power is absolutely continuous with respect to the Haar measure thus solving a problem posed by Andrzej Hulanicki.

The idea used in the proof of the theorem is taken from [1], [2]. In fact we prove a more general

THEOREM. *The convolution of two radial measures from $M(\boldsymbol{H}^n)$, $n \geqslant 2$, or three radial measures from $M(\boldsymbol{H}^1)$, which do not have mass in the center of $\boldsymbol{H}^n$ is absolutely continuous with respect to the Haar measure.*

First we prove the following

LEMMA. *Let G be a locally compact, separable topological group, and let K be a subset of $M(G)$ which is convex, compact in the $*$-weak topology. Then for every k, every $\mu_1, \ldots, \mu_k \in K$ and for a Borel set B we have*

$$(2.1) \quad \mu_1 * \ldots * \mu_k(B) = \int_{\mathrm{ext}K} \ldots \int_{\mathrm{ext}K} w_1 * \ldots * w_k(B)\, dM_1(w_1) \ldots dM_k(w_k)$$

where $M_1, \ldots, M_k$ are measures which correspond to $\mu_1, \ldots, \mu_k$ via Choquet

Theorem, *i.e.*,

$$\mu_i(B) = \int_{\operatorname{ext} K} w(B)\,dM_i(w). \tag{2.2}$$

Proof. For every $i = 1, 2, \ldots, k$ and a Borel set B let

$$f(x_i) = \int_G \ldots \int_G \chi_B(x_1 \cdot \ldots \cdot x_k)\,d\mu_1(x_1) \ldots d\mu_{i-1}(x_{i-1})\,d\mu_{i+1}(x_{i+1}) \ldots d\mu_k(x_k).$$

Then, by (2.2),

$$\begin{aligned}\mu_1 * \ldots * \mu_k(B) &= \int_G f(x_i)\,d\mu_i(x_i)\\ &= \int_{\operatorname{ext} K} w(f)\,dM_i(w)\\ &= \int_{\operatorname{ext} K} \mu_1 * \ldots * w * \ldots * \mu_k(B)\,dM_i(w),\end{aligned}$$

and thus (2.1) follows by an easy induction. □

Proof of the theorem. Of course we may consider only probability measures. Let K^n be the set of radial, probability measures in H^n. The extreme points of K^n are of two kinds: the normalized rotation invariant measures $\mu_{r,t}$ on the spheres $(S^{2n-1}(r), t) \subseteq H^n$, where $S^{2n-1}(r)$ is the sphere of radius r with the center at the origin in R^{2n}, and the atomic measures δ_t, $t \in Z^n$. Let us denote $\Delta_n = \{\delta_t\colon t \in Z^n\}$ and let $\mu_1, \ldots, \mu_k \in K^n$ be the measures with no mass in Z^n. If we take the characteristic function of Z^n then from (2.2) we get $M_i(\Delta_n) = 0$, i.e., M_i is concentrated outside the set Δ_n. Hence (2.1) turns into

$$\mu_1 * \ldots * \mu_k(B) = \int_{\operatorname{ext} K \setminus \Delta_n} \ldots \int_{\operatorname{ext} K \setminus \Delta_n} w_1 * \ldots * w_k(B)\,dM_1(w_1) \ldots dM_k(w_k). \tag{2.3}$$

Now we consider the two cases separately.

Case 1. $n = 1$. According to (2.3), since $\mu_{r,t} = \mu_r * \delta_t$ and δ_t are in the center of $M(H^1)$, it suffices to show that $\mu_p * \mu_q * \mu_r$ is absolutely continuous, where $\mu_s = \mu_{s,0}$ is the normalized Lebesgue measure concentrated on the circle $\{(s\sin\varphi, s\cos\varphi, 0)\colon 0 \leqslant \varphi < 2\pi\}$. For $p, q, r > 0$ we define the mapping

$$h\colon S^1 \times S^1 \times S^1 \to H^1$$

given by

$$h(\varphi, \psi, \theta) = (p\cos\varphi, p\sin\varphi, 0)(q\cos\psi, q\sin\psi, 0)(r\cos\theta, r\sin\theta, 0).$$

In coordinates (φ, ψ, θ) the jacobian of the mapping h is an analytic function. It is easy to see that at the point $(0, 0, \frac{1}{2}\pi)$ we have $\dfrac{\partial h}{\partial(\varphi, \psi, \theta)} \neq 0$. Since the set of zeros of a nonzero analytic function has measure zero, then the mapping h has rank 3 except on the set D of measure zero. Then for a set $B \subseteq H^1$ of measure zero we have

$$\mu_p * \mu_q * \mu_r(B) = \int_{S^1 \times S^1 \times S^1} \chi_B(h(\varphi, \psi, \theta)) dm(\varphi, \psi, \theta) = m(h^{-1}(B)) = m(h^{-1}(B) - D), \tag{2.4}$$

where m is the Lebesgue measure on $S^1 \times S^1 \times S^1$. From the implicit function theorem it follows that for every point at which h has rank 3 there exist local coordinates in which h is the identity. Since the measure m on $S^1 \times S^1 \times S^1$ is equivalent to the Lebesgue measure on any coordinate patch, then, from (2.4) we deduce that $\mu_p * \mu_q * \mu_r (B) = 0$. Hence $\mu_p * \mu_q * \mu_r$ is absolutely continuous.

Case 2. $n \geqslant 2$. Let μ_r be the normalized Lebesgue surface measure on the sphere $S^{2n-1}(r)$. As in the case $n = 1$, it is sufficient to show that the convolution of two measures μ_p, μ_q on H^n is absolutely continuous. For $p, q > 0$ we define a mapping h: $S^{2n-1} \times S^{2n-1} \to \boldsymbol{R}$ by

$$h(x, y) = (px, 0) \cdot (qy, 0).$$

We are going to show that h has rank $2n+1$ everywhere except $\omega_{2n-1} \times \times \omega_{2n-1}$-null set D, where ω_{2n-1} is the normalized surface measure on S^{2n-1}. As in the case $n = 1$ it suffices to show that there exists a point at which h has rank $2n+1$. Consider the standard parametrization near the point $((0, \dots, 0, 1), (0, \dots, 0, 1)) \in S^{2n-1} \times S^{2n-1}$ given by

$$x = \Big(x_1, \dots, x_{2n-1}, \big(1 - \sum_{i=1}^{2n-1} x_i^2\big)^{1/2}\Big), \qquad y = \Big(y_1, \dots, y_{2n-1}, \big(1 - \sum_{i=1}^{2n-1} y_i^2\big)^{1/2}\Big).$$

An easy computation shows that in these coordinates the jacobian of h has rank $2n+1$ at the point (x, y) where $x = (0, \dots, 0, 1)$ and $y_2 = \dots = y_{2n-1} = 0$ and y_1 sufficiently small. For a Borel set B of Lebesgue measure zero on H^n

$$\mu_p * \mu_q(B) = \int_{S^{2n-1}} \int_{S^{2n-1}} \chi_B(h(x, y)) d\omega_{2n-1}(x) d\omega_{2n-1}(y) = \omega_{2n-1} \times \omega_{2n-1}(h^{-1}(B)) = \omega_{2n-1} \times \omega_{2n-1}(h^{-1}(B) - D).$$

But at every point of $S^{2n-1} \times S^{2n-1} - D$ the implicit function theorem says that after a suitable change of coordinates h is an orthogonal projection. By the Fubini Theorem, the inverse image under a projection of a Lebesgue null set is a null set and the measures $\omega_{2n-1} \times \omega_{2n-1}$ and the Lebesgue measure in these coordinates are equivalent so $\mu_p * \mu_q(B) = 0$. Thus $\mu_p * \mu_q$ is absolutely continuous. □

Remarks. Consider three dimensional Heisenberg group. Let μ be the normalized Lebesgue measure on the circle $(\cos\varphi, \sin\varphi, 0)$ and ν the Lebesgue measure on $(0, 0, t)$, $0 \leqslant t \leqslant 1$. Since ν is a central measure we have

$$(\mu+\nu)^{*n} = \sum_{k=0}^{n} \binom{n}{k} \nu^{*(n-k)} * \mu^{*k}.$$

For $k \geqslant 3$ each summand is absolutely continuous. An argument as above shows that $\nu^{*(n-2)} * \mu^{*2}$ is absolutely continuous. But for $k = 0, 1$ we have $\operatorname{supp} \nu^{*n} \subseteq (0, 0, \boldsymbol{R})$ and $\operatorname{supp} \nu^{*(n-1)} * \mu \subseteq \{(\cos\varphi, \sin\varphi, t)\colon 0 \leqslant t \leqslant n\}$. Thus every convolution power of $\mu+\nu$ has a singular part outside the center.

REFERENCES

[1] D. L. Ragozin, *Rotation invariant measures on Euclidean spaces*, Indiana University Mathematical Journal 23 (1974), p. 1139 1154.

[2] –, *Central measures on compact simple Lie groups*, Journal of Functional Analysis 10 (1972), p. 212–229.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
WROCŁAW, POLAND

Reçu par la Rédaction le 9.01.1981