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A FORMAL ANALOGY BETWEEN PROXIMITY AND FINITE DIMENSIONALITY

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0. Introduction. In looking for classes of uniform spaces similar to proximity spaces, one might proceed as follows. Two subsets A and B of a uniform space fail to be proximal iff $\{A, B\}$ is uniformly discrete. Thus the proximity structure (δ-structure) of a uniform space is determined by its finite uniformly discrete families. Analogously, one may define the Δ-structure $\Delta\mu$ of μX to consist of all uniformly discrete families of μX.

The category of Δ-spaces so obtained is isomorphic with the reflexive closure (in the category of uniform spaces) of the class of all finite-dimensional uniform spaces. On the other hand, Δ-spaces can be defined *via* a formal axiomatization similar to that given for proximity spaces. Some basic properties of Δ-spaces are given, along with results comparing δ-spaces with Δ-spaces.

1. Finite dimensionality. When not otherwise indicated, the notation here is like that in Isbell [2]. It will be convenient to consider a uniform space μX as consisting of a set X together with a Tukey uniformity μ on X.

Definition 1. A *Δ-space* is a set X together with a collection $\Delta^*\mu$ consisting of all uniformly discrete families of subsets of a uniform space μX. Δ^* is the mapping $\mu X \mapsto \Delta^*\mu X$. A *$\Delta^*$-function* is a function $f\colon \Delta^*\mu X \to \Delta^*\nu Y$ such that $\forall \mathscr{V} \in \Delta^*\nu$, $\{f^{-1}[V]\colon V \in \mathscr{V}\} \in \Delta^*\mu$.

By a *discrete family* we shall always mean a uniformly discrete family. A collection $\mathscr{P}$ *refines* a collection $\mathscr{Q}$ if $\forall P \in \mathscr{P}$, $\exists Q \in \mathscr{Q}$; $P \subseteq Q$. A cover $\mathscr{U}$ of a set X has *dimension n* if

$$\exists U_0, \ldots, U_n \in \mathscr{U};\ U_0 \cap \ldots \cap U_n \neq \emptyset$$

and

$$\forall V_0, \ldots, V_{n+1} \in \mathscr{U};\ V_0 \cap \ldots \cap V_{n+1} = \emptyset.$$

For any uniform space X, a collection $\mathscr{Q}$ is a *strict shrinking* of a collection $\mathscr{P}$ of subsets of X if there is a 1-1 map $\lambda\colon \mathscr{Q} \to \mathscr{P}$ and a $\mathscr{W} \in \mu$ such that $\forall Q \in \mathscr{Q}$, $\mathscr{W} * Q \subseteq \lambda(Q)$.

We shall repeatedly need the following facts about a uniform space from Isbell [2]. Every uniform covering has a uniform strict shrinking (Proposition 19, p. 65). Every n-dimensional uniform cover has a uniform star refinement of dimension not greater than n (this is a direct consequence of Corollary 12, p. 62). Every n-dimensional uniform cover has a uniform refinement which is the union of $n+1$ discrete families (by Proposition 25, p. 67).

Definition 2. A uniform space is a *Δ-uniform space* if its uniformity has a base of finite-dimensional covers.

THEOREM 1. (a) *For each uniform space μX, the set of all zero- and one-dimensional uniform covers is a subbase for a uniformity $\Delta\mu$.*

(b) *The finite-dimensional covers of μ form a base for $\Delta\mu$.*

(c) *The map $\mu X \mapsto \Delta\mu X$ is a reflection onto the subcategory of Δ-uniform spaces.*

(d) *The class of Δ-uniform spaces is the reflexive closure of all finite--dimensional uniform spaces.*

Proof. (a) Let μX be a uniform space. Let β be the set of all zero- and one-dimensional uniform covers of μX. Using induction, we see that each $\mathscr{U}_0 \in \beta$ is the largest element in some nested sequence of star refinements in β. So β is the union of a set of countable bases for zero- and one--dimensional uniformities, and whence it is a subbase for a subproduct of one-dimensional uniform spaces.

(b) It is clear that $\Delta\mu$ has a base of finite-dimensional covers. But suppose $\mathscr{U}$ is any n-dimensional uniform cover of μX; we need to show that $\mathscr{U} \in \Delta\mu$. Let $\mathscr{V}$ be a uniform refinement of $\mathscr{U}$ which is the union of $n+1$ discrete families $\mathscr{P}_0, \ldots, \mathscr{P}_n$. Let $\mathscr{W}$ be a strict shrinking of $\mathscr{V}$; then $\mathscr{W}$ is, clearly, also the union of $n+1$ discrete families $\mathscr{Q}_0, \ldots, \mathscr{Q}_n$, with each $\mathscr{Q}_i$ a strict shrinking of $\mathscr{P}_i$. For each i, let

$$\mathscr{X}_i = \mathscr{P}_i \cup \{X - \bigcup \mathscr{Q}_i\};$$

$\mathscr{X}_i$ is, obviously, a uniform cover of X which is (at most) one-dimensional. Then $\mathscr{X}_0 \cap \ldots \cap \mathscr{X}_n$ is a refinement of $\mathscr{V}$, and, therefore, of $\mathscr{U}$, since

$$(X - \bigcup \mathscr{Q}_0) \cap \ldots \cap (X - \bigcup \mathscr{Q}_n) = \emptyset.$$

Hence $\mathscr{U} \in \Delta\mu$.

(c) To see that the map $\mu X \mapsto \Delta\mu X$ is a reflection, we need to show that, for any uniform spaces μX and νY, a function $f\colon \mu X \to \Delta\nu Y$ is uniformly continuous iff $f\colon \Delta\mu X \to \Delta\nu Y$ is. But this follows directly from the fact that the inverse image of a one-dimensional cover is at most one-dimensional.

(d) Suppose Λ is any reflector which is the identity on finite-dimensional uniform spaces. Then, given μX and any finite-dimensional uniformity $\nu \subseteq \mu$, the following diagram of identity maps must commute:

$$\begin{tikzcd} \mu X \arrow[r] \arrow[d] & \Lambda\mu X \arrow[d] \\ \nu X \arrow[r] & \nu X \end{tikzcd}.$$

Consequently, $\forall \mu$, $\Lambda\mu \supseteq \Delta\mu$. But then $\forall \mu$, $\Lambda\Delta\mu \supseteq \Delta\Delta\mu = \Delta\mu \supseteq \Lambda\Delta\mu$. This shows (d).

THEOREM 2. *The map Δ^*: $\Delta\mu X \mapsto \Delta^* \Delta\mu X = \Delta^* \mu X$ is an isomorphism from the category of Δ-uniform spaces onto the category of Δ-spaces.*

Proof. First, we show that Δ^* is onto. It suffices to prove that, for any uniformity μ, $\Delta^* \mu = \Delta^* \Delta\mu$. Since $\mu \supseteq \Delta\mu$, we have $\Delta^* \mu \supseteq \Delta^* \Delta\mu$. On the other hand, suppose $\mathscr{A} \in \Delta^* \mu$; pick $\mathscr{U} \in \mu$ so that $\mathscr{U} * \mathscr{A}$ is disjoint. Let

$$\mathscr{V} = \mathscr{U} * \mathscr{A} \cup \{X - \bigcup \mathscr{A}\}.$$

Then $\mathscr{V}$ is uniform since it is refined by $\mathscr{U}$. Now, $\mathscr{V} \in \Delta\mu$, and $\mathscr{V} * \mathscr{A} = \mathscr{U} * \mathscr{A}$ is disjoint. Hence $\mathscr{A} \in \Delta^* \Delta\mu$.

Next, we show that Δ^* is 1-1. Given $\Delta^* \mu X$, we must be able to determine what $\Delta\mu X$ was. Let Γ be the collection of all families of the form $\mathscr{Q} \cup \{X - \bigcup \mathscr{P}\}$, where $\mathscr{P}$ refines $\mathscr{Q}$, and $\mathscr{Q}$ and $\mathscr{P} \cup \{X - \bigcup \mathscr{Q}\}$ belong to $\Delta^* \mu X$. It suffices to show that Γ is a subbase for $\Delta\mu$.

First, we show that each element of Γ is a zero- or one-dimensional μ-uniform cover of X. Indeed, let $\mathscr{Q} \cup \{X - \bigcup \mathscr{P}\}$ be a typical element of Γ; it obviously covers X and is at most one-dimensional. To see $\mathscr{Q} \cup \{X - \bigcup \mathscr{P}\}$ is uniform, let $\mathscr{U} \in \mu$ be such that $\mathscr{U} * \mathscr{Q}$ and $\mathscr{U} * [\mathscr{P} \cup \{X - \bigcup \mathscr{Q}\}]$ are disjoint. Pick $U \in \mathscr{U}$. If $U \cap \bigcup \mathscr{P} = \emptyset$, then $U \subseteq X - \bigcup \mathscr{P}$. Otherwise, U meets some $P \in \mathscr{P}$, and whence some $Q \in \mathscr{Q}$. But then U meets only that element of $\mathscr{Q}$ since $\mathscr{U} * \mathscr{Q}$ is disjoint; U fails to meet $X - \bigcup \mathscr{Q}$ since $\mathscr{U} * [\mathscr{P} \cup \{X - \bigcup \mathscr{Q}\}]$ is disjoint; hence $U \subseteq Q$. Consequently, $\mathscr{U}$ refines $\mathscr{Q} \cup \{X - \bigcup \mathscr{P}\}$.

Second, to see that Γ does generate $\Delta\mu$, let $\mathscr{U}$ be a one-dimensional cover in $\Delta\mu$. As in the previous proof, we let $\mathscr{P} \cup \mathscr{P}' \in \Delta\mu$ be a refinement of $\mathscr{U}$ such that $\mathscr{P}$ and $\mathscr{P}'$ are discrete, and let $\mathscr{Q} \cup \mathscr{Q}' \in \Delta\mu$ be a strict shrinking of $\mathscr{P} \cup \mathscr{P}'$, with $\mathscr{Q}$ and $\mathscr{Q}'$ strict shrinkings of $\mathscr{P}$ and $\mathscr{P}'$, respectively; so that

$$(\mathscr{P} \cup \{X - \bigcup \mathscr{Q}\}) \cap (\mathscr{P}' \cup \{X - \bigcup \mathscr{Q}'\})$$

is a uniform refinement of $\mathscr{U}$ which is the intersection of two elements of Γ.

It remains to show that a function f: $\Delta\mu X \to \Delta\nu Y$ is uniformly continuous iff f: $\Delta^*\mu X \to \Delta^*\nu Y$ is a Δ-function. If f is uniformly continuous, then, of course, the inverse image of any discrete family is discrete. Conversely, if f is a Δ-map, and Γ_μ and Γ_ν are subbases like the one constructed above, then the inverse image of each element of Γ_ν is, clearly, in Γ_μ; hence f is uniformly continuous.

Remark 1. One would guess that results for finite-dimensional uniform spaces tend to carry over to Δ-spaces. One such example is that, for any Δ-uniformity μ, $\Delta d\mu = \delta d\mu$. This is shown in the proof of Isbell's Theorem V.5, p. 79 of [2].

2. Axiomatic characterization. The purpose of the following discussion is to find a characterization of Δ-spaces in terms of properties which are simple and intuitively reasonable. Of the following, (4) is perhaps least expected; however, some such condition seems necessary.

THEOREM 3. *A collection η of disjoint families of subsets of X is a Δ-structure iff it satisfies the following conditions:*

(1) $\emptyset \in \eta$; *if* $A \subseteq X$, *then* $\{A\} \in \eta$.

(2) *If* $\mathscr{A}, \mathscr{B} \in \eta$, *then*

$$\mathscr{A} \wedge \mathscr{B} = \{A \cap B\colon A \in \mathscr{A}, B \in \mathscr{B}\} \in \eta.$$

(3) *If* $\mathscr{A} \in \eta$ *and* γ *is a partition of* $\mathscr{A}$, *then*

$$\mathscr{A} | \gamma = \left\{\bigcup \mathscr{G}\colon \mathscr{G} \in \gamma\right\} \in \eta.$$

(4) *If* $\mathscr{A} | M, \mathscr{A} | N, \{M - N, N - M\} \in \eta$, *then* $\mathscr{A} | M \cup N \in \eta$.

(5) *If* $\mathscr{A} \in \eta$, *then there is a* $\mathscr{B} \in \eta$ *such that* $\left\{\bigcup \mathscr{A}, X - \bigcup \mathscr{B}\right\} \in \eta$ *and* $\mathscr{A}$ *is a* 1-1 *refinement of* $\mathscr{B}$, *in the sense that* $\mathscr{B} | \bigcup \mathscr{A} = \mathscr{A}$.

Before giving a proof we will need a definition and a lemma.

Definition 3. For any family $\mathscr{A}$, let

$$[\mathscr{A}] = \bigcup \{A \times B\colon A, B \in \mathscr{A}, \text{ and } A \neq B\}.$$

LEMMA 1. *If η is a collection of disjoint families of subsets of X, then conditions* (1)-(4) *are equivalent to*

(6) *If Γ is a finite subset of η and* $[\mathscr{A}] \subseteq \bigcup \{[\mathscr{P}]\colon \mathscr{P} \in \Gamma\}$, *then* $\mathscr{A} \in \eta$.

Proof. First, suppose (6) is true.

(1) $[\emptyset] = \emptyset$, and if $A \subseteq X$, then $[\{A\}] = \emptyset$; but

$$\forall \Gamma \subseteq \eta,\ \emptyset \subseteq \bigcup \{[P]\colon P \in \Gamma\}.$$

(2) If $\mathscr{A}, \mathscr{B} \in \eta$, then $[\mathscr{A} \wedge \mathscr{B}] \subseteq [\mathscr{A}] \cup [\mathscr{B}]$.

(3) If $\mathscr{A} \in \eta$ and γ is a partition of $\mathscr{A}$, then $[\mathscr{A} | \gamma] \subseteq [\mathscr{A}]$.

(4) If $M, N \subseteq X$, and $\mathscr{A} | M, \mathscr{A} | N, \{M - N, N - M\} \in \eta$, then

$$[\mathscr{A} | M \cup N] \subseteq [\mathscr{A} | M] \cup [\mathscr{A} | N] \cup [\{M - N, N - M\}].$$

Now suppose conditions (1)-(4) hold.

(I) If $\mathscr{A}, \{\bigcup \mathscr{A}\} \cup \mathscr{B} \in \eta$, then $\mathscr{A} \cup \mathscr{B} \in \eta$.

Indeed, if $\mathscr{A}, \{\bigcup \mathscr{A}\} \cup \mathscr{B} \in \eta$, then $\{\bigcup \mathscr{A}, \bigcup \mathscr{B}\} \in \eta$, by (3) and the fact that $\bigcup \mathscr{A}$ and $\bigcup \mathscr{B}$ are disjoint. By (1) and (2),

$$\mathscr{B} = (\{\bigcup \mathscr{A}\} \cup \mathscr{B}) \wedge \{\bigcup \mathscr{B}\} \in \eta.$$

Hence, by (4), $\mathscr{A} \cup \mathscr{B} \in \eta$.

(II) If $\mathscr{F}$ is a finite collection of subsets of X and $\forall A, B \in \mathscr{F}$, $\mathscr{A} | A \cup B \in \eta$, then $\mathscr{A} | \bigcup \mathscr{F} \in \eta$. Indeed, if $\mathscr{F} = \emptyset$, then, by (1), $\mathscr{A} | \bigcup \mathscr{F} = \emptyset \in \eta$. Suppose the statement is true whenever $\mathscr{F}$ has n elements. Let $\mathscr{F}$ have $n+1$ elements $F_0, \ldots, F_n$, and let $\forall A, B \in \mathscr{F}$, $\mathscr{A} | A \cup B \in \mathscr{F}$. Then, by hypothesis,

$$\mathscr{A} | F_0 \cup \ldots \cup F_{n-1} \in \eta \quad \text{and} \quad \mathscr{A} | F_1 \cup \ldots \cup F_n \in \eta.$$

Let

$$M = \bigcup \mathscr{A} \cap (F_0 \cup \ldots \cup F_{n-1}) \quad \text{and} \quad N = \bigcup \mathscr{A} \cap (F_1 \cup \ldots \cup F_n).$$

Now, $\mathscr{A} | (F_0 \cup F_n) \in \eta$, and thus, by (1) and (2),

$$\mathscr{A} | M \Delta N = \mathscr{A} | (F_0 \cup F_n) \wedge \{M \Delta N\} \in \eta,$$

so that, by (3), $\{M - N, N - M\} \in \eta$. Hence, by (4), $\mathscr{A} | F_0 \cup \ldots \cup F_n \in \eta$.

If Γ is empty, the hypothesis of (6) reduces to $[\mathscr{A}] \subseteq \emptyset$, in which case $\mathscr{A}$ has at most one element, and thus, by (1), belongs to η.

Next, suppose that $[\mathscr{A}] \subseteq [\mathscr{P}]$ and $\mathscr{P} \in \eta$; then $\mathscr{A} \in \eta$. Indeed, if $\mathscr{A}$ has at most one element, then, by (1), $\mathscr{A} \in \eta$. So assume $\mathscr{A}$ has more than one element, in which case $[\mathscr{A}] \subseteq [\mathscr{P}]$ implies $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{P}$. Also,

$$\mathscr{P} | \bigcup \mathscr{A} = \mathscr{P} \wedge \{\bigcup \mathscr{A}\} \in \eta \quad \text{and} \quad [\mathscr{A}] \subseteq [\mathscr{P} | \bigcup \mathscr{A}],$$

so we can assume that $\bigcup \mathscr{P} = \bigcup \mathscr{A}$. For each $A \in \mathscr{A}$, let

$$\mathscr{G}_A = \{P \in \mathscr{P};\ P \cap A \neq \emptyset\}.$$

Notice that each $P \in \mathscr{P}$ meets at most one element of $\mathscr{A}$ since $[\mathscr{A}] \subseteq [\mathscr{P}]$; hence it is contained in some element of $\mathscr{A}$ since $\bigcup \mathscr{P} \subseteq \bigcup \mathscr{A}$. Consequently, $\gamma = \{\mathscr{G}_A : A \in \mathscr{A}\}$ is a partition of $\mathscr{P}$ and $\forall A \in \mathscr{A}$, $\bigcup \mathscr{G}_A = A$. Hence $\mathscr{A} = \mathscr{P} | \gamma \in \eta$.

Now, for $n > 1$, suppose that (6) has been shown whenever Γ has less than n elements, $\Gamma = \{\mathscr{P}_1, \ldots, \mathscr{P}_n\} \subseteq \eta$, and $[\mathscr{A}] \subseteq [\mathscr{P}_1] \cup \ldots \cup [\mathscr{P}_n]$. It will be convenient to use the following notation: for each family denoted by a script letter, the corresponding Latin letter with a tilda will denote its union; for example, $\tilde{K} = \bigcup \mathscr{K}$. The first step in verifying (6) for Γ is the following

(III) If $\mathscr{A} | \tilde{P}_i \cap \bigcup \{\tilde{P}_j : j \neq i\} \in \eta$ for some i, then $\mathscr{A} | \tilde{P}_i \in \eta$.

Indeed, let

$$\mathscr{H} = \{H \in \mathscr{A} \wedge \mathscr{P}_i : H \subseteq \bigcup \{\tilde{P}_j : i \neq j\}\} \quad \text{and} \quad \mathscr{K} = \mathscr{A} \wedge \mathscr{P}_i - \mathscr{H}.$$

Since

$$[\mathscr{H}] \subseteq [(\mathscr{A} | \tilde{P}_i \cap \bigcup \{\tilde{P}_j : j \neq i\}) \wedge \mathscr{P}_i],$$

we have $\mathscr{H} \in \eta$.

Next, we can show that $\{\tilde{H}\} \cup \mathscr{K} \in \eta$ by showing that $[\{\tilde{H}\} \cup \mathscr{K}] \subseteq [\mathscr{P}_i]$. Suppose $x \in A \cap P \in \mathscr{K}$ and $y \in A' \cap P' \in \mathscr{A} \wedge \mathscr{P}_i$, where $A \cap P$ and $A' \cap P'$ are distinct. We need to show that $(x, y) \in [\mathscr{P}_i]$. Either $A \neq A'$ or $P \neq P'$; if $P \neq P'$, we are done. If $A \neq A'$, we can choose an $x' \in A \cap P - \bigcup \{\tilde{P}_j : i \neq j\}$ since $A \cap P \in \mathscr{K}$. Then

$$(x', y) \in [\mathscr{A}] \subseteq [\mathscr{P}_1] \cup \ldots \cup [\mathscr{P}_n],$$

but x' belongs only to $\tilde{\mathscr{P}}_i$, and thus $(x', y) \in [\mathscr{P}_i]$, in which case $P \neq P'$. Therefore, by step (I), $\mathscr{H} \cup \mathscr{K} = \mathscr{A} \wedge \mathscr{P}_i \in \eta$. Finally, $[\mathscr{A} | \tilde{P}_i] \subseteq [\mathscr{A} \wedge \mathscr{P}_i]$, and thus $\mathscr{A} | \tilde{P}_i \in \eta$.

(IV) If $\mathscr{A} | \tilde{P}_i \in \eta$ for each i, then $\mathscr{A} \in \eta$.

Indeed, let Π be the partition of $\bigcup \bigcup \Gamma$ consisting of all non-empty sets of the form $Q_1 \cap \ldots \cap Q_n$, where, for each i, either $Q_i = \tilde{P}_i$ or $Q_i = \bigcup \bigcup \Gamma - \tilde{P}_i$. Notice that, for each i and each $K \in \Pi$, either $K \cap \tilde{P}_i = \emptyset$ or $K \subseteq \tilde{P}_i$. Using step (II), we can show that $\mathscr{A} | \bigcup \Pi = \mathscr{A} \in \eta$. Pick $J, K \in \Pi$. If $\tilde{A} \cap (J \cup K) \subseteq \tilde{P}_i$ for some i, then $[\mathscr{A} | J \cup K] \subseteq [\mathscr{A} | \tilde{P}_i]$, and thus $\mathscr{A} | J \cup K \in \eta$. On the other hand, if $\tilde{A} \cap (J \cup K)$ is not contained in any $\tilde{P}_i$, then $\mathscr{A} | J \cup K$ has at most one element. Suppose not; then we can choose $A, A' \in \mathscr{A} | J \cup K$ so that A meets J and A' meets K. Pick $x \in A \cap J$ and $x' \in A' \cap K$; then

$$(x, x') \in [\mathscr{A} | J \cup K] \subseteq [\mathscr{P}_1] \cup \ldots \cup [\mathscr{P}_n],$$

so that some $\tilde{P}_i$ meets $A \cap J$ and $A' \cap K$, in which case $J \cup K \subseteq \tilde{P}_i$, contrary to the assumption. Therefore, by step (II), $\mathscr{A} | \bigcup \Pi = \mathscr{A} \in \eta$.

Before finishing the induction argument, we need to consider a couple of special cases. First, suppose Γ has two elements, $\mathscr{P}$ and $\mathscr{Q}$. Then

$$[\mathscr{A} | \tilde{P} \cap \tilde{Q}] \subseteq [\mathscr{P} \wedge \mathscr{Q}],$$

and thus, by step (III), $\mathscr{A} | \tilde{P}, \mathscr{A} | \tilde{Q} \in \eta$, so that, by step (IV), $\mathscr{A} \in \eta$. Next, suppose Γ has three elements $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$. Then

$$[\mathscr{A} | \tilde{P} \cap \tilde{Q}] \subseteq [\mathscr{P} \wedge \mathscr{Q}] \cup [\mathscr{R}],$$

and thus $\mathscr{A}|\tilde{P} \cap \tilde{Q} \in \eta$ by induction; similarly, $\mathscr{A}|\tilde{P} \cap \tilde{R} \in \eta$. We want to show that $\mathscr{A}|\tilde{P} \cap (\tilde{Q} \Delta \tilde{R}) \in \eta$, so that, by step (I), we will have $\mathscr{A}|\tilde{P} \cap (\tilde{Q} \cup \tilde{R}) \in \eta$, where

$$\mathscr{F} = \{\tilde{P} \cap \tilde{Q} \cap \tilde{R}, \tilde{P} \cap (\tilde{Q} - \tilde{R}), \tilde{P} \cap (\tilde{R} - \tilde{Q})\},$$

and thus, by step (III), we have $\mathscr{A}|\tilde{P} \in \eta$. By symmetry, $\mathscr{A}|\tilde{Q}$ and $\mathscr{A}|\tilde{R}$ will also belong to η, and thus, by step (IV), so will $\mathscr{A}$. To verify that $\mathscr{A}|\tilde{P} \cap (\tilde{Q} \Delta \tilde{R}) \in \eta$, let

$$F = \tilde{P} \cap (\tilde{Q} \Delta \tilde{R}), \qquad \mathscr{H} = \{H \in \mathscr{A} \wedge \mathscr{P}|F\colon H \cap \tilde{R} = \emptyset\},$$

and

$$\mathscr{K} = \mathscr{A} \wedge \mathscr{P}|F - \mathscr{H}.$$

We first show that $\mathscr{H}, \{\tilde{H}\} \cup \mathscr{K} \in \eta$, so that, by step I, $\mathscr{H} \cup \mathscr{K} = \mathscr{A} \wedge \mathscr{P}|F \in \eta$. $\mathscr{H} \subseteq \mathscr{A} \wedge \mathscr{P}|(\tilde{Q} - \tilde{R})$, so that

$$[\mathscr{H}] \subseteq [\mathscr{A} \wedge \mathscr{P}|\tilde{Q}] \subseteq [\mathscr{A} \wedge \mathscr{P} \wedge \mathscr{Q}] \subseteq [\mathscr{P} \wedge \mathscr{Q}] \cup [\mathscr{R}],$$

and $\mathscr{H} \in \eta$. To see that $\{\tilde{H}\} \cup \mathscr{K} \in \eta$, we show that $[\{\tilde{H}\} \cup \mathscr{K}] \subseteq [\mathscr{P}] \cup [\mathscr{R}]$. Suppose $x \in A \cap P \cap F \in \mathscr{K}$ and $y \in A' \cap P' \cap F \in \mathscr{A} \wedge \mathscr{P}|F$ with $A \cap P \cap F$ distinct from $A' \cap P' \cap F$; then either $A \neq A'$ or $P \neq P'$. If $P \neq P'$, we are done. Suppose $A \neq A'$. If $x, y \in \tilde{R}$, then $x, y \notin \tilde{Q}$; and since

$$(x, y) \in [\mathscr{A}] \subseteq [\mathscr{P}] \cup [\mathscr{Q}] \cup [\mathscr{R}],$$

(x, y) must belong to $[\mathscr{P}] \cup [\mathscr{R}]$. On the other hand, if $y \notin \tilde{R}$, we can choose an $x' \in A \cap P \cap F \cap \tilde{R}$ since $A \cap P \cap F \notin \mathscr{H}$. Then

$$(x', y) \in [\mathscr{A}] \subseteq [\mathscr{P}] \cup [\mathscr{Q}] \cup [\mathscr{R}];$$

but $x' \notin \tilde{Q}$ and $y \notin \tilde{R}$, so that $(x', y) \in [\mathscr{P}]$, and thus $P \neq P'$. Therefore, $[\{\tilde{H}\} \cup \mathscr{K}] \subseteq [\mathscr{P}] \cup [\mathscr{R}]$, and $\mathscr{A} \wedge \mathscr{P}|F \in \eta$. Since $[\mathscr{A}|F] \subseteq [\mathscr{A} \wedge \mathscr{P}|F]$, we conclude that

$$\mathscr{A}|F = \mathscr{A}|\tilde{P} \cap (\tilde{Q} \Delta \tilde{R}) \in \eta.$$

Consequently, $\mathscr{A} \in \eta$.

Finally, to finish the argument, assume $n \geqslant 4$. Let $\mathscr{P}, \mathscr{Q}_1, \mathscr{Q}_2$ and $\mathscr{R}$ be four different elements of Γ. Since

$$[\mathscr{A}|\tilde{P} \cap \tilde{Q}_i] \subseteq [\mathscr{P} \wedge \mathscr{Q}_i] \cup \bigcup \{[\mathscr{P}_k]\colon \mathscr{P}_k \neq \mathscr{P}, \mathscr{Q}_i\},$$

$\mathscr{A}|\tilde{P} \cap \tilde{Q}_i \in \eta$ for $i = 1, 2$. We can show that $\mathscr{A}|\tilde{P} \cap (\tilde{Q}_1 \Delta \tilde{Q}_2) \in \eta$ as follows. For $i = 1, 2$, let

$$J_i = \tilde{P} \cap (\tilde{Q}_1 \Delta \tilde{Q}_2) \cap (\tilde{Q}_i - \tilde{R}), \qquad K_i = \tilde{P} \cap (\tilde{Q}_1 \Delta \tilde{Q}_2) \cap (\tilde{Q}_i \cap \tilde{R}).$$

Then, one can immediately see that, for $i, j = 1, 2,\ i \neq j$,

$$[\mathscr{A}|J_1 \cup J_2] \subseteq \bigcup \{[\mathscr{P}_k]:\ \mathscr{P}_k \neq \mathscr{R}\},$$
$$[\mathscr{A}|K_1 \cup K_2] \subseteq [\mathscr{P} \wedge \mathscr{R}] \cup \bigcup \{[\mathscr{P}_k]:\ \mathscr{P}_k \neq \mathscr{P}, \mathscr{R}\},$$
$$[\mathscr{A}|J_i \cup K_i] \subseteq [\mathscr{P} \wedge \mathscr{Q}_i] \cup \bigcup \{[\mathscr{P}_k]:\ \mathscr{P}_k \neq \mathscr{P}, \mathscr{Q}_i\},$$
$$[\mathscr{A}|J_i \cup K_j] \subseteq [\mathscr{A}|J_i] \cup [\mathscr{A}|K_j] \cup \bigcup \{[\mathscr{P}_k]:\ \mathscr{P}_k \neq \mathscr{Q}_i, \mathscr{Q}_j, \mathscr{R}\}.$$

Induction and the first two inclusions tell us that $\mathscr{A}|J_i$ and $\mathscr{A}|K_i$ belong to η. This, induction, and all four inclusions imply that $\mathscr{A}$, restricted to any two of the sets J_1, J_2, K_1 and K_2, belongs to η. Step (II) gives $\mathscr{A}|\tilde{P} \cap (\tilde{Q}_1 \Delta \tilde{Q}_2) \in \eta$, and then step (I) gives $\mathscr{A}|\tilde{P} \cap (\tilde{Q}_1 \cup \tilde{Q}_2) \in \eta$. By symmetry, for distinct indices i, j, k, we have $\mathscr{A}|\tilde{P}_i \cap (\tilde{P}_j \cup \tilde{P}_k) \in \eta$. Hence, by steps (II), (III) and (IV), $\mathscr{A}|\tilde{P}_i \cap \bigcup \{\tilde{P}_j:\ j \neq i\} \in \eta$, $\mathscr{A}|\tilde{P}_i \in \eta$ for each i and, finally, $\mathscr{A} \in \eta$.

Proof of Theorem 3. For this proof it will be convenient to use Weil-uniformities (see Kelley [3]) in place of the usual Tukey-uniformities. First, suppose ηX is a Δ-space given by a (Weil)-uniformity $\mathscr{U}$. We need to show that conditions (5) and (6) are satisfied. Pick an $\mathscr{A} \in \eta$, and choose $U, V \in \mathscr{U}$ so that $\{U[A]:\ A \in \mathscr{A}\}$ is disjoint, and $V \circ V \subseteq U$. Let $\mathscr{B} = \{V[A]:\ A \in \mathscr{A}\}$. Then $\{V[B]:\ B \in \mathscr{B}\}$ is disjoint, so that $\mathscr{B} \in \eta$, and $\{\bigcup \mathscr{A},\ X - \bigcup \mathscr{B}\} \in \eta$ since $V[\bigcup \mathscr{A}] = \bigcup \mathscr{B}$. This shows (5).

For (6), suppose $[\mathscr{A}] \subseteq [\mathscr{P}_1] \cup \ldots \cup [\mathscr{P}_n]$ with each $\mathscr{P}_i \in \eta$. For each i, choose $U_i \in \mathscr{U}$, so that $\{U_i[P]:\ P \in \mathscr{P}_i\}$ is disjoint. Then, it is easy to see that $\{(U_1 \cap \ldots \cap U_n)[A]:\ A \in \mathscr{A}\}$ is disjoint, so that $\mathscr{A} \in \eta$.

Now suppose ηX satisfies conditions (5) and (6). For each $\mathscr{A} \in \eta$, let $U(\mathscr{A}) = X \times X - [\mathscr{A}]$, and let $\mathscr{W} = \{U(\mathscr{A}):\ \mathscr{A} \in \eta\}$. We show that $\mathscr{W}$ is a subbase for a (Weil)-uniformity $\mathscr{U}$ for which η is the set of all $\mathscr{U}$-discrete families on X. Pick $U(\mathscr{A}) \in \mathscr{W}$; by (5), we can choose a $\mathscr{B} \in \eta$, so that $\mathscr{A}$ is a 1-1 refinement of $\mathscr{B}$ and $\{\bigcup \mathscr{A}, X - \bigcup \mathscr{B}\} \in \eta$. Let

$$W = U(\mathscr{B}) \cap U(\{\bigcup \mathscr{A}, X - \bigcup \mathscr{B}\});$$

then $W \circ W \subseteq U(\mathscr{A})$. Indeed, suppose that $(x, y), (y, z) \in W$, but that $(x, z) \notin U(\mathscr{A})$. Then $(x, z) \in [\mathscr{A}]$ and we can choose $A, A' \in \mathscr{A}$ and $B, B' \in \mathscr{B}$, so that $A \neq A'$, $x \in A \subseteq B$, and $z \in A' \subseteq B'$. Now, (x, y) belongs to W and, therefore, it does not belong to $[\mathscr{B}]$ or $[\{\bigcup \mathscr{A}, X - \bigcup \mathscr{B}\}]$. From $x \in A \in \mathscr{A}$ and $(x, y) \notin [\{\bigcup \mathscr{A}, X - \bigcup \mathscr{B}\}]$, we have $y \notin X - \bigcup \mathscr{B}$, so that $y \in \bigcup \mathscr{B}$. Then $x \in B$ and $(x, y) \notin [\mathscr{B}]$ give $y \in B$. Similarly, $(y, z) \in W$ and $z \in A' \subseteq B'$ give $y \in B'$, contrary to the disjointness of $\mathscr{B}$. Therefore, $W \circ W \subseteq U(\mathscr{A})$. Hence $\mathscr{W}$ is a subbase for a uniformity $\mathscr{U}$.

To see that η is the collection of all $\mathscr{U}$-discrete families, notice, first, that if $A, A' \in \mathscr{A} \in \eta$ with $A \neq A'$, then $U(\mathscr{A})[A] \cap A' = \emptyset$; this is sufficient to make $\mathscr{A}$ a $\mathscr{U}$-discrete family. Conversely, suppose $\mathscr{A}$ is a $\mathscr{U}$-discrete family. Then we can choose

$$W = U(\mathscr{A}_1) \cap \dots \cap U(\mathscr{A}_n) \in \mathscr{U} \quad \text{with } \mathscr{A}_1, \dots, \mathscr{A}_n \in \eta,$$

so that $\{W[A]: A \in \mathscr{A}\}$ is disjoint. Pick $A, A' \in \mathscr{A}$ with $A \neq A'$. Take $x \in A$ and $y \in A'$; then $(x, y) \notin W$, so that $(x, y) \notin U(\mathscr{A}_i)$ for some $\mathscr{A}_i$, and thus $(x, y) \in [\mathscr{A}_i]$. This shows that $\mathscr{A} \subseteq [\mathscr{A}_1] \cup \dots \cup [\mathscr{A}_n]$. Hence, by (6), $\mathscr{A} \in \eta$.

3. Basic properties. δ-spaces have the property that two disjoint subspaces A and B of a space X fail to be proximal iff $A \cup B$ is isomorphic to the (direct) sum $A \oplus B$. Consequently, δ-spaces are simply a description of the finite sum structure of uniform spaces. For Δ-spaces, however, the situation is not so nice. The functor Δ does not commute with sums. But Δ-spaces do have sums, and Δ^* does commute with them.

THEOREM 4. *Δ^* commutes with sums.*

Proof. Let $\{\mu_a X_a\}$ be a collection of uniform spaces. We can assume that $\{X_a\}$ is disjoint and that $\bigcup \{X_a\}$ is the underlying set for a direct sum $\oplus\{\mu_a X_a\}$. Then it is easy to see that a family $\mathscr{A}$ of disjoint subsets of X belongs to $\Delta^* \oplus \{\mu_a X_a\}$ or to $\oplus\{\Delta^* \mu_a X_a\}$ iff $\mathscr{A} | X_a \in \Delta^* \mu_a X_a$ for each a.

THEOREM 5. *Suppose Γ is an infinite collection of uniform spaces; then $\oplus\, \Gamma$ is a Δ-uniform space iff there is an integer m such that each $\mu X \in \Gamma$ has dimension not greater than m. Consequently, Δ and $\oplus$ do not commute.*

Proof. Assume the spaces in Γ are disjoint. First, suppose we can choose a sequence $\{\mu_j X_j: j \in \omega\}$ from Γ so that each $\mu_j X_j$ has dimension not less than j. Then, we can, of course, find a cover for $\oplus\, \Gamma$ such that $\mathscr{A} | X_j$ for each $j \in \omega$ fails to have a uniform refinement of dimension less than j. Consequently, $\mathscr{A}$ fails to have a finite-dimensional uniform refinement, and thus it is not in $\Delta \oplus \Gamma$. Next, if, for some m, each $\mu X \in \Gamma$ has dimension not greater than m, then $\oplus\, \Gamma$, clearly, has dimension not greater than m.

The class of subproducts of one-dimensional uniform spaces is, obviously, closed under products, so that Δ-spaces have products. Δ, however, does not commute with products.

LEMMA 2. *For any uniform spaces μX and νY, $\Delta(\mu \times \nu)$ is stronger than $\Delta\mu \times \Delta\nu$.*

Proof. $\Delta\mu \times \Delta\nu$ has a subbase of covers of the form

$$\mathscr{P} \times \mathscr{Q} = \{P \times Q: P \in \mathscr{P}, Q \in \mathscr{Q}\},$$

where $\mathscr{P}$ and $\mathscr{Q}$ are finite-dimensional uniform covers of $\Delta\mu$ and $\Delta\nu$, respectively. But each such $\mathscr{P} \times \mathscr{Q}$, clearly, belongs to $\Delta(\mu \times \nu)$.

PROPOSITION 1. *If νY is totally bounded, then $\Delta(\mu X \times \nu Y) = \Delta\mu X \times \Delta\nu Y$.*

Proof. Suppose νY is totally bounded; then $\Delta\nu = \nu$ and we need only show that $\Delta\mu\times\nu$ contains $\Delta(\mu\times\nu)$. Suppose that $\mathscr{A}$ is a $\Delta(\mu\times\nu)$-discrete family. Choose $\mathscr{P}\times\mathscr{Q}\in\mu\times\nu$ so that $(\mathscr{P}\times\mathscr{Q})*\mathscr{A}$ is disjoint and $\mathscr{Q}$ is finite. For each $A\in\mathscr{A}$ and $Q\in\mathscr{Q}$, let

$$P_{QA} = \bigcup\{P\in\mathscr{P}\colon P\times Q\cap A\neq\emptyset\},\quad P_Q = \bigcup\{P\in\mathscr{P}\colon P\times Q\cap\bigcup\mathscr{A}=\emptyset\},$$

$$\mathscr{P}_Q = \{P_Q\}\cup\{P_{QA}\colon A\in\mathscr{A}\},$$

and let $\mathscr{P}^\wedge$ be the intersection of the $\mathscr{P}_Q$'s. Each $\mathscr{P}_Q$ is at most one-dimensional and is refined by $\mathscr{P}$. $\mathscr{P}^\wedge$ is the finite intersection of the $\mathscr{P}_Q$'s and, therefore, belongs to $\Delta\mu$. Furthermore, $(\mathscr{P}^\wedge\times\mathscr{Q})*\mathscr{A}$ is disjoint; to see this suppose $\mathscr{Q}=\{Q_1,\ldots,Q_n\}$ and pick an $A\in\mathscr{A}$. Then, any element of $\mathscr{P}^\wedge\times\mathscr{Q}$ which meets A has the form $(P_{Q_1A}\cap\ldots\cap P_{Q_nA})\times Q$ for some $Q\in\mathscr{Q}$, and

$$(P_{Q_1A}\cap\ldots\cap P_{Q_nA})\times Q\subseteq P_{QA}\times Q = (\mathscr{P}\times\{Q\})*A\subseteq(\mathscr{P}\times\mathscr{Q})*A.$$

Therefore, $\mathscr{A}\in\Delta\mu\times\nu$.

Example 1. *Let μX be a direct sum of $\{\boldsymbol{R}^n\colon n\in\omega\}$, where $\boldsymbol{R}^n$ is a Euclidean n-space. Let νY be ω with the discrete uniformity. Then we have $\Delta(\mu X\times\nu Y)\neq\Delta\mu X\times\Delta\nu Y$.*

Proof. Assume $\{\boldsymbol{R}^n\colon n\in\omega\}$ is disjoint. For each $n\in\omega$, let $\mathscr{U}_n$ be a uniform cover of $\boldsymbol{R}^n$ which fails to have a refinement of dimension less than n, and is the intersection of n one-dimensional uniform covers $\mathscr{P}_{1n},\ldots,\mathscr{P}_{nn}$. Let $\mathscr{U}=\bigcup\{\mathscr{U}_n\colon n\in\omega\}$; then $\mathscr{U}$ belongs to μ and does not have a finite--dimensional uniform refinement. For each j, let

$$\mathscr{P}_j = \bigcup\{\{\boldsymbol{R}^n\}\colon n<j\}\cup\{\mathscr{P}_{jn}\colon n\geqslant j\}.$$

Then each $\mathscr{P}_j$ is a one-dimensional uniform cover of μX and

$$\mathscr{U} = \bigcap\{\mathscr{P}_j\colon j\in\omega\}.$$

Let $\mathscr{V}=\{P\times\{i\}\colon i\in Y, P\in\mathscr{P}_i\}$; then $\mathscr{V}$ is a one-dimensional cover of $X\times Y$ refined by $\mathscr{U}\times\{\{i\}\colon i\in Y\}$, and whence belongs to $\Delta(\mu\times\nu)$. But $\mathscr{V}$ cannot belong to $\Delta\mu\times\nu$ since it would be then refined by a cover of the form $\mathscr{W}\times\{\{i\}\colon i\in Y\}$, where $\mathscr{W}$ is a finite-dimensional uniform cover of μX. This would mean $\mathscr{W}$ refined each $\mathscr{P}_i$, and whence $\mathscr{U}$, which is impossible.

Results for δ-spaces similar to the above theorem and example are given in Isbell [2], Exercise 12, p. 34. Isbell uses them to show that two δ-equivalent uniform spaces need not have a δ-equivalent least upper bound. I do not know whether the corresponding statement for Δ-equivalent uniform spaces is true. (**P 902**)

Since the completion functor preserves dimension and commutes with products, the class of all subproducts of one-dimensional uniform spaces is closed under completion. Thus completeness for Δ-spaces can be taken to be the same as completeness for the corresponding uniform spaces. The following question seems interesting:

If μX is complete, under what conditions (if any) is $\Delta\mu X$ *not* complete? The problem can be related to another functor λ using a version of a theorem by Shirota. For any uniform space μX, the *locally fine* co-reflection $\lambda\mu$ of μ is the weakest uniformity stronger than μ such that, for every covering $\mathscr{A}$ of X, if $\exists \mathscr{U}\in\lambda\mu, \forall U\in\mathscr{U}, \mathscr{A}|U\in\lambda\mu$, then $\mathscr{A}\in\lambda\mu$. Ginsberg and Isbell [1] have generalized Shirota's theorem to state that if μ is locally fine, complete and non-measurable, then $c\mu$, the weak uniformity induced by all real-valued continuous functions on μX, is complete.

PROPOSITION 2. *Suppose μX is non-measurable and complete, and that $\Delta\lambda\mu = \lambda\Delta\mu$. Then $\Delta\mu X$ is complete.*

Proof. If μ is complete, then $\lambda\mu$, being stronger than μ, is also. Thus $c\lambda\mu$ is complete by Shirota's theorem. $\Delta\lambda\mu = \lambda\Delta\mu$ is likewise complete, as it is stronger than $c\lambda\mu$. But $\lambda\Delta\mu$ is complete iff $\Delta\mu$ is, as a result of Proposition 12 of [2], p. 127. Therefore, $\Delta\mu$ is complete.

A nice property of the total boundedness is that it is preserved by uniformly continuous functions. The extent to which the Δ-uniform property behaves this way is suggested by the following two results. The proof of the first generalizes the example on p. 79 of [2] of a space for which $\Delta d\mu X > \delta d\mu X$.

LEMMA 3. *For any uniformities μ and ν, if $\mu\wedge\delta\nu$ is a Δ-uniformity, then so is μ.*

Proof. Pick $\mathscr{P}\in\mu$. Let $\mathscr{Q}\in\mu\wedge\delta\nu$ be a finite-dimensional refinement of $\mathscr{P}$. Let $\{F_i\colon i\leqslant k\}\in\delta\nu$ and $\{V_a\colon a\in\Gamma\}\in\mu$ be such that $\{V_a\}<^*\mathscr{P}$ and $\{F_i\}\wedge\{V_a\}$ refines $\mathscr{Q}$. Then, by Theorem IV. 20 of [2], p. 66, we can, for each $i\leqslant k$, let $\{U_{ia}\}$ be an isomorphic extension of $\{V_a\}|F_i$ over a μ-uniform neighborhood of F_i in a way such that

$$\forall\beta\in\Gamma,\ U_{i\beta}\subseteq\{V_a\}*(V_\beta\cap F_i).$$

Let $\mathscr{W} = \{U_{ia}\colon i\leqslant k, a\in\Gamma\}$. Then $\mathscr{W}<\mathscr{P}$, since each $U_{i\beta}$ is contained in some element of $\{V_a\}*\{V_a\}$, and $\{V_a\}<^*\mathscr{P}$; $\mathscr{W}$ is also the finite union of finitely many finite-dimensional families. Finally, by Lemmas V. 4 and V. 3 of [2], p. 79, $\mathscr{W}\in\mu$. Hence μ is a Δ-uniformity.

THEOREM 6. *If every uniformity between $\Delta\mu$ and $\delta\mu$ is a Δ-uniformity, then $\Delta\mu = \delta\mu$.*

Proof. Suppose $\Delta\mu\neq\delta\mu$. Let D be an infinite discrete subset of $\Delta\mu X$, where $X = \bigcup\bigcup\mu$. Choose $\mathscr{U}_0, \mathscr{U}_0'\in\Delta\mu$ so that $\mathscr{U}_0'<\mathscr{U}_0$ and $\mathscr{U}_0*\{\{x\}\colon x\in D\}$ is a discrete subfamily of $\mathscr{U}_0$. Let $\mathscr{A}_0 = \mathscr{U}_0'*\{\{x\}\colon x\in D\}$. Having chosen $\mathscr{U}_n$, $\mathscr{U}_n'$, and $\mathscr{A}_n$, take $\mathscr{U}_{u+1}, \mathscr{U}_{n+1}'\in\Delta\mu$ and $\mathscr{A}_{n+1}$ so that

$$\mathscr{A}_{n+1} = \mathscr{U}_{n+1}'*\{\{x\}\colon x\in D\}\subseteq\mathscr{U}_{n+1}\quad\text{and}\quad\mathscr{U}_{n+1}'<\mathscr{U}_{n+1}<^*\mathscr{U}_n'.$$

For each $n \in \omega$, let $\{\mathscr{I}_{nk}: k \in \omega\}$ be a star-nested base for the unit n-cube I^n with an essential n-dimensional cover $\mathscr{I}_{n0}$. For each $n \in \omega$, let $J_n \subseteq I^n$ contain just one point in each non-empty subset of the form $\bigcap \mathscr{H} \cap \bigcap (\mathscr{I}_{nn} - \mathscr{H})$ for $\mathscr{H} \subseteq \mathscr{I}_{nn}$. By induction on n, let f_n be a 1-1 correspondence from a finite subset of

$$\{A \in \mathscr{A}_n: \forall m < n, A \cap \bigcup \operatorname{dom} f_m = \emptyset\}$$

onto J_n. For each $m \in \omega$, let

$$\mathscr{W}_m = \mathscr{U}_m \cup \{\bigcup \check{f}_n [H]: H \in \mathscr{I}_{nm}, n \geqslant m\}.$$

It is clear that, for each m and $n \geqslant m+1$,

$$\mathscr{U}_{m+1} * \{\bigcup \check{f}_n [H]: H \in \mathscr{I}_{n,m+1}\} \quad \text{and} \quad \{\bigcup \check{f}_n [H]: H \in \mathscr{I}_{n,m+1}\} * \mathscr{U}_{m+1}$$

are refinements of $\{\bigcup \check{f}_n [H]: H \in \mathscr{I}_{nm}\}$. Consequently, $\mathscr{W}_{m+1} <^* \mathscr{W}_m$. Let ν be the uniformity generated by $\{\mathscr{W}_m: m \in \omega\}$. Now, ν is not a Δ-uniformity since if $\mathscr{W} \in \nu$ with $\mathscr{W}_m < \mathscr{W} < \mathscr{W}_0$, then $\forall n \geqslant m$, $\mathscr{W} | \bigcup \check{f}_n [I^n]$ is n-dimensional since $\mathscr{I}_{n0}$ is an essential n-dimensional cover. Thus $\mathscr{W}$ is infinite-dimensional. By Lemma 3, $\nu \wedge \delta\mu$ is not a Δ-uniformity either. Finally, $\delta\mu \subseteq \nu \wedge \delta\mu \subseteq \Delta\mu$.

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