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AN ISOPERIMETRIC BOUND FOR A SOBOLEV CONSTANT

BY

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1. Introduction. In a recent paper [1], the author studied the problem of finding a constant Ω^2 for the Sobolev-type inequality

$$\int_D (\psi_i\psi_i)^2\,dx \leqslant \frac{1}{\Omega^2}\Big\{\int_D \psi_{i,j}\psi_{i,j}\,dx\Big\}^2 \qquad (i, j = 1, 2, 3),$$

where $\psi_i(x)$ are sufficiently smooth functions which are defined on a bounded three-dimensional domain D and vanish on the boundary of D. Here we are employing the summation convention and a comma denotes differentiation. It was shown in [1] that

$$\Omega^2 \geqslant 23\pi\left[\frac{4\pi}{3V}\right]^{1/3},$$

where V denotes the volume of D. In the present work we will sharpen this inequality for Ω^2 and in the process get an isoperimetric inequality for the first eigenvalue of a certain non-linear eigenvalue problem. The techniques presented here are applicable to the computation of other isoperimetric inequalities for the first eigenvalues in a class of non-linear eigenvalue problems (see [2]).

2. New lower bound. It was shown in [1] (and hence, we will not repeat those arguments here) that computing a value for Ω^2 is related to calculating a lower bound for the variationally characterized eigenvalue

$$\bar{\Omega}^2 = \inf_{u\in\Gamma(D)} \frac{\Big\{\int_D u_{,j}u_{,j}\,dx\Big\}^2}{\int_D u^4\,dx}, \tag{1}$$

where $\Gamma(D)$ denotes the space of Dirichlet integrable, scalar-valued functions which are defined on D and vanish on ∂D. In particular, we will be interested in finding a constant Ω^2 (which depends only on the geometry of D) such that $\Omega^2 \leqslant \bar{\Omega}^2$.

The associated Euler system for (1) is given by

(2a) $$\Delta u+\bar{\lambda}u^3 = 0 \quad \text{in } D,$$

(2b) $$u = 0 \quad \text{on } \partial D,$$

where $\bar{\lambda} = \bar{\Omega}^2/B(u)$. Here we have introduced the following notation for the Dirichlet integral of u:

$$B(u) = \int\limits_D u_{,j}u_{,j}\,dx.$$

As in [1], we consider a special case of (1). Namely, let D_R denote a sphere of radius R, and let r be the radial coordinate in spherical coordinates (r, θ, φ). We then consider the variational problem

(3) $$\Omega_R^2 = \inf_{u\in\Gamma_r(D_R)} \frac{\Big\{\int\limits_{D_R} u_{,j}u_{,j}\,dx\Big\}^2}{\int\limits_{D_R} u^4\,dx},$$

where $\Gamma_r(D_R) = \{u \in \Gamma(D_R): u = u(r)\}$. In this special case, (2) reduces to

(4a) $$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}+\lambda u^3 = 0, \qquad r \in [0, R),$$

(4b) $$u(R) = \frac{du}{dr}(0) = 0,$$

where

(5) $$\lambda = \frac{\Omega_R^2}{B(u)}.$$

We will be interested in the first eigenvalue and eigenfunction of this eigenvalue problem.

In the previous paper we derived a "Rellich-type" identity for solutions of (4):

(6) $$\int\limits_{D_R} u^4\,dx = \left[\frac{B(u)}{2\pi R}\right]^{1/2}\int\limits_{D_R} u^3\,dx.$$

Also we showed that (2) is related to an Emden-Fowler boundary-value problem by the change of dependent and independent variables:

(7a) $$u(r) = u(0)y(z),$$

(7b) $$z = r\sqrt{\lambda}\,u(0).$$

In particular, we obtain

$$\frac{d^2y}{dz^2}+\frac{2}{z}\frac{dy}{dz}+y^3 = 0$$

such that $y(0) = 1$ and $y'(0) = 0$. If z_0 denotes the first positive zero of $y(z)$, then, matching boundary conditions with $u(R) = 0$, by (7b) we have

$$R\sqrt{\lambda}u(0) = z_0. \tag{8}$$

The "Rellich-type" identity and interrelation between the Euler system for (3) and the Emden-Fowler boundary-value problem will play the leading roles in our computation of Ω_R^2.

Returning to (4a), by a direct integration of the ordinary differential equation we get

$$u'(R) = -\frac{\lambda}{R^2}\int_0^R u^3 r^2 dr.$$

Changing the one-dimensional integral into a volume integral and using (5) and (6), we obtain

$$u'(R) = -\frac{\Omega_R^2}{2R\sqrt{2\pi R}[B(u)]^{3/2}}\int_{D_R} u^4 dx. \tag{9}$$

Since

$$\Omega_R^2 \int_{D_R} u^4 dx = [B(u)]^2,$$

we can calculate from (9) the representation for the Dirichlet integral of u:

$$B(u) = 8\pi R^3 [u'(R)]^2. \tag{10}$$

On the other hand, we can calculate from (8) an alternative representation

$$B(u) = \left[\frac{Ru(0)}{z_0}\right]^2 \Omega_R^2, \tag{11}$$

and hence, equating (10) and (11), we find

$$\Omega_R^2 = 8\pi R z_0^2 \left[\frac{u'(R)}{u(0)}\right]^2. \tag{12}$$

The difficulty here is that we do not have the necessary information for the first eigenfunction of (4) at $r = 0, R$. However, we can relate $u(r)$ and $y(z)$ by transformations (7). Namely, a straightforward calculation yields

$$u'(R) = \frac{\Omega_R [u(0)]^2 y'(z_0)}{\sqrt{B(u)}}. \tag{13}$$

Using (13) in (12) and substituting for $u(0)$ the expression derived from (8), we finally have

$$\Omega_R^2 = \frac{8\pi z_0^4 [y'(z_0)]^2}{R}. \tag{14}$$

The utility of the representation of Ω_R^2 given in (14) is that it involves only information about the solution of a second-order initial-value problem which we can easily handle numerically to compute z_0 and $y'(z_0)$. Using the Runge-Kutta-Fehlberg scheme (see [3]) to integrate the Emden--Fowler initial-value problem, it was found that

$$z_0 \doteq 6.8969 \quad \text{and} \quad y'(z_0) \doteq -0.0424,$$

which gives a lower bound for Ω_R^2:

$$\Omega_R^2 \geqslant \frac{32\pi}{R}. \tag{15}$$

It should be emphasized that (14) is an exact expression for Ω_R^2 and (15) is given only for comparison purposes.

Having found an expression for $\bar{\Omega}^2$ in the special case where D is a sphere of radius R (i.e., Ω_R^2), we repeat the symmetrization arguments of [1] to conclude for the general case that

$$\bar{\Omega}^2 \geqslant \frac{8\pi z_0^2 [y'(z_0)]^2}{R} > \frac{32\pi}{R},$$

where R is the radius of a sphere having the same volume as D. Equivalently, we have

$$\bar{\Omega}^2 \geqslant \Omega^2 = [8\pi z_0^4 [y'(z_0)] \left[\frac{4\pi}{3V}\right]^{1/3} > 32 \left[\frac{4\pi}{3V}\right]^{1/3} \tag{16}$$

or

$$\int_D (\psi_i \psi_i)^2 \, dx \leqslant \frac{1}{32\pi} \left[\frac{3V}{4\pi}\right]^{1/3} \left\{ \int_D \psi_{i,j} \psi_{i,j} \, dx \right\}^2.$$

3. Conclusion. We remark that the first inequality in (16) for Ω^2 is isoperimetric, since equality holds for a sphere.

Using the techniques presented above and in [1], one can derive other isoperimetric inequalities for eigenvalues. For example, if D is now a two-dimensional bounded domain and λ is the smallest eigenvalue of the eigenproblem for fixed $p = 1, 2, \ldots,$

$$\Delta\varphi + \lambda\varphi^{2p+1} = 0 \text{ in } D, \quad \varphi = 0 \text{ on } \partial D,$$

then in [2] it is shown that

$$\lambda \geqslant \frac{1}{R^2} [\pi(p+1)]^p z_0^{2p+2} [y'(z_0)]^{2p}, \tag{17}$$

where R is the radius of a disk having the same area as D, and $y(z)$ is the solution of the equation $zy'' + y' + zy^{2p+1} = 0$ with $y(0) = 1$ and $y'(0) = 0$, z_0 being the first positive zero of $y(z)$. Inequality (17) is isoperimetric.

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