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ON A PROBLEM CONCERNING UNIFORM LIMITS OF DARBOUX FUNCTIONS

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The uniform limits of Darboux functions has been investigated by Bruckner, Ceder and Weiss (1). It was shown that the class of real-valued Darboux functions of Baire class 1 defined on an interval, is closed under uniform limits and a problem was posed whether the class of Darboux functions in Baire class 2 is closed under uniform limits (see also P 546). In the present paper it is shown that this class fails to be closed.

THEOREM. *There is a sequence* $\{f_n\}_{n=1}^{\infty}$ *of Darboux functions of Baire class* 2, *defined on the open unit interval* $I = (0, 1)$, *and converging uniformly to a function* f *which fails to satisfy the Darboux condition on* I.

Proof: Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint nowhere dense non-empty perfect subsets of I such that each open subinterval of I contains some P_n (such a sequence does exist; see e.g. the proof of Lemma 4.1 in (1)). For each n, the set of component intervals of $I-P_n$ with natural ordering is similar to the set of rational numbers contained in the closed unit interval $J = \langle 0, 1\rangle$; let φ_n be a corresponding isomorphism. Define functions g_n' as follows: If G is some component of $I-P_n$ and $x\in G$ let $g_n'(x) = \varphi_n(G)$; if $x\in P_n$ let $g_n'(x) = \inf\{g_n'(y);\ y\in I-P_n \text{ and } y > x\}$. Now put $g_n = g_n'|P_n$. It is easy to see that each g_n maps the set P_n continuously onto J and that g_n takes on each irrational value from J exactly at one point of P_n. Let ξ be some irrational number in J; for each n, let a_n be the point of P_n for which $g_n(a_n) = \xi$. Define functions f_n and f as follows:

$$f_n(x) = \begin{cases} \max\left\{\xi-\dfrac{1}{m}, 0\right\} & \text{if } x = a_m \text{ and } m \leqslant n, \\ g_m(x) & \text{if } x\in P_m \text{ and } x \neq a_1, \ldots, a_n, \\ 0 & \text{otherwise} \end{cases}$$

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(1) A. M. Bruckner, J. G. Ceder and M. Weiss, *Uniform limits of Darboux functions*, Colloquium Mathematicum 15 (1966), p. 65 - 77.

$$f(x) = \begin{cases} \max\left\{\xi - \dfrac{1}{m}, 0\right\} & \text{if } x = a_m, \text{ for some } m, \\ g_m(x) & \text{if } x \in P_m \text{ and } x \neq a_1, a_2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that the functions f_n and f have all desired properties (for each $x \in I$, $f(x) \neq \xi$).

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