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WEAKLY ALMOST PERIODIC MAPPINGS ON ONE- AND TWO-MANIFOLDS

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1. Introduction and notation. This paper is a continuation of [1], thus we remind only the most important terms.

The mapping φ on a compact space X is *weakly almost periodic* (w.a.p.) provided the semigroup $S_\varphi = \operatorname{cl}\{\varphi, \varphi^2, \ldots\}$ consists only of continuous mappings (cl denotes the point convergence closure), and it is *strongly almost periodic* (s.a.p.) provided φ has equicontinuous iterates. By M we will denote the union of all minimal invariant sets for the system (X, φ). It is known that for a w.a.p. system there exists a retraction $\varepsilon \in S_\varphi$ of X onto M, and that φ is s.a.p. on M (see [4] and [1]). We will also investigate the set M_2 of all recurrent points, which is a superset of M (see [1]); a point $x \in X$ is *recurrent* if $x = \psi(x)$ for some $\psi \in S_\varphi$. The system (X, φ) is called *transitive* if there is a point $x_0 \in X$ for which $\{x_0\} \cup S_\varphi(x_0) = X$. It follows from [1] that if X is connected, then for a w.a.p. transitive system (X, φ), φ is a homeomorphism, M is a single minimal set, and every point $x \in X$ is recurrent.

We give now some general lemma which will be used in the last part of this paper.

LEMMA. *Let x be a recurrent point for a system (X, φ). Then x is recurrent also for the system (X, φ^n), where $n \in N$.*

Proof. It is clear that for each open set $U_0 \ni x$ there exist a natural number n_1 and an open set $U_1 \ni x$, $U_1 \subset U_0$, such that $\varphi^{n_1}(U_1) \subset U_0$. By induction we obtain a sequence n_k of natural numbers, and a decreasing sequence U_k of neighbourhoods of x with the property $\varphi^{n_k}(U_k) \subset U_{k-1}$ for each k. Then $\varphi^m(x) \in U_0$ for every m which may be obtained as a sum of a few numbers n_k with different indices k. Now it remains to apply an easy number-theoretic lemma which says that for any sequence n_k and an arbitrary $n \in N$ there exists m of the above form divisible by n. Thus we have found in U_0 a point from $S_{\varphi^n}(x)$. Now, since $S_{\varphi^n}(x)$ is closed, the assertion follows.

2. One-dimensional case.

THEOREM 1. *Let I be the unit interval and let $\varphi\colon I \to I$ be a continuous mapping. The following conditions are equivalent:*

(i) *φ is w.a.p.*

(ii) *The set $D = \{x \in I\colon \varphi^2(x) = x\}$ is an interval (or a point).*

(iii) *φ^{2n} converges uniformly.*

(iv) *φ is s.a.p.*

Proof. (i) $\Rightarrow$ (ii). By (i) the set M is an interval (as a retract of I) and, by the proof of Theorem 3.2 in [4], $\varphi^2|M$ is the identity. So $M = D$, and (ii) is proved.

(ii) $\Rightarrow$ (iii). The condition (ii) may be expressed as follows: the intersection K of the graph G of φ with its reflection $s(G)$ around the main diagonal is connected. Now suppose that $I \neq D$ and φ is essential. Thus G is an arc joining all the sides of the square $I \times I$. Clearly, $s(G)$ has the same property and the arcs do not intersect the diagonal out of K. Suppose that K does not attain any of the sides. Then some component of $G \setminus K$ joins K with the lower and right sides below the diagonal. So does one of the components of $s(G) \setminus K$. Thus the above components intersect each other, which is a contradiction. If now K reaches some of the sides, by symmetry, it also reaches the symmetric side. Consider, e.g., the left and lower sides. Since $K \neq G$, K does not attain the right and upper sides. Now $G \setminus K$ is an arc joining K with the upper and right sides, whence it intersects the diagonal. The obtained contradictions prove that if $D \neq I$, then φ is not essential. Consider the sequence of intervals $\varphi^n(I)$. It decreases to an invariant interval on which φ is essential, whence, by the above calculation, the limit set is D. Thus we have $\varphi^n(x) \to D$ on I. Since $\varphi^2|D$ is the identity, it is easy to estimate that φ^{2n} converges uniformly, as (iii) demands.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious.

THEOREM 2. *Let C be the circle. The continuous mapping $\varphi\colon C \to C$ is w.a.p. if and only if one of the following conditions holds:*

(a) *φ is a rotation or a reflection.*

(b) *φ is inessential and it is w.a.p. on the interval $\varphi(C)$.*

(c) *The system (C, φ) is a factor (by the identification $0 = 1$) of some system $(I, \hat{\varphi})$ on the unit interval, for which $\hat{\varphi}(x) = x \Leftrightarrow x = 0$ or 1.*

In the cases (a) *and* (b), *φ is also s.a.p., and in the case* (c) *it is not s.a.p.*

Proof. It is well known that (a) $\Rightarrow$ s.a.p. By Theorem 1 also (b) $\Rightarrow$ s.a.p. Suppose that (c) holds. The whole graph of $\hat{\varphi}$ lies below or above the diagonal (except for 0 and 1), so $\hat{\varphi}^n(y) \to 0$ or 1 for all $y \in I$. Thus $\varphi^n(x) \to x_0$ for all $x \in C$, where x_0 is the image of the identification $0 = 1$, whence φ is w.a.p. In this case the iterates φ^n are easily seen not to be equicontinuous, so φ is not s.a.p. Conversely, suppose that φ is w.a.p. If it is not essential, the condition (b) holds. If it is essential, then, by Theorem 3.2 in [4], the set M is equal to C or it reduces to a single fixed point x_0. If $M = C$, then φ being s.a.p. is well known to be a rotation or a reflection. It remains to show $M = \{x_0\} \Rightarrow$(c). First observe that the condition (c) may be expressed as follows: there exists a single fixed point $x_0 \in C$, and if we open the circle in x_0, we do not disconnect the graph of φ; more precisely, for any closed interval $J \subset C$ not containing x_0, the image $\varphi(J)$ does not contain x_0 in its interior. Now, suppose that $M = \{x_0\}$ and (c) is not true, i.e., there exists a closed interval $J \subset C$, $J \not\ni x_0$, with $x_0 \in \operatorname{int} \varphi(J)$. Then there is a neighbourhood U of x_0 contained in $\varphi(J)$ and disjoint from J. By continuity of φ there exists also an open interval $V \ni x_0$, $V \subset U$, with $\varphi(V) \subset U$. Since φ is essential, we have $\varphi(V^c) \supset U^c$, and since $V^c \supset J$, also $\varphi(V^c) \supset U$. Thus $\varphi(V^c) = C$; in particular, $\varphi(V^c) \supset V^c$. Write

$$H_0 = V^c \quad \text{and} \quad H_1 = \varphi^{-1}(H_0) \cap H_0.$$

Let $x \in H_0$. Since $H_0 \subset \varphi(H_0)$, there exists $y \in H_0$ for which $\varphi(y) = x$. Now $\varphi(y) = x \in H_0$, and hence $y \in \varphi^{-1}(H_0)$. Thus $y \in H_1$, and we have proved $H_0 \subset \varphi(H_1)$. Conversely, for any $x \in \varphi(H_1)$ we have $x = \varphi(y)$ for some $y \in H_1 \subset \varphi^{-1}(H_0)$, and thus $x = \varphi(y) \in H_0$, whence $H_0 = \varphi(H_1)$. Obviously, $H_1 \subset H_0$. By induction, the sequence of closed sets $H_{n+1} = \varphi^{-1}(H_n) \cap H_n$ has the property

$$H_n = \varphi(H_{n+1}) \supset H_{n+1} \quad \text{for each } n \geqslant 0.$$

Observe that $\varphi(H_1) = H_0 \neq \emptyset$ implies $H_1 \neq \emptyset$, and inductively $H_n \neq \emptyset$. Hence the set

$$H = \bigcap_n H_n$$

is nonempty, closed and invariant, so it contains a minimal set. We have obtained a contradiction, since $M = \{x_0\}$ and $x_0 \notin H_0 \supset H$.

3. Two-dimensional case. For w.a.p. systems on two-manifolds Montgomery et al. have given in [5] a complete classification of the center system $(M, \varphi|M)$. However, we can still say not much about the action of φ on the whole manifold. For example, we do not even know if there may exist recurrent points out of M (which is impossible for one-manifolds; this topic has been investigated in [1]). In particular, if φ is transitive and w.a.p., then each point of the manifold is recurrent. Our main result, which follows now, is proving that in this case the system is minimal.

THEOREM 3. *Each w.a.p. transitive system (X, φ) on a compact two-manifold is minimal.*

Before proving the theorem we give some algebraic-topological lemmas (see [3] for terms and details).

LEMMA 1. *Let Γ be a continuum contained in a compact connected two-manifold X without boundary. Denote by U_n some sequence of closed metric* *balls around Γ of diameters decreasing to zero (thus $U_{n+1} \subset U_n$). Then, for m and n sufficiently large,*

$$\tau(\pi(U_n)) = \tau(\pi(U_m)),$$

where $\pi(U_n)$ denotes the fundamental group of U_n, and τ is the homomorphism of fundamental groups induced by the embedding into X.

Proof. (A) Let $C_1, C_2, \ldots, C_n$ be disjoint open submanifolds of dimension two in X, such that $X \setminus \bigcup_{i=1}^{n} C_i$ is connected. Then no more than m of C_i are different from the disc, where m is defined as the number of prime factors in the unique representation of X as a connected sum of tori or projective planes. We omit the easy proof of this fact.

(B) Attaching to U_n all the discs which are components of the complement U_n^c of U_n we obtain a compact connected CW-complex V_n with $\pi(V_n) = \tau(\pi(U_n))$. By connectedness of U_n, each component of U_n^c contained in a disc-component of U_{n+1}^c is also a disc, whence $V_{n+1} \subset V_n$ follows for each n.

(C) We will show that $V_2 \cup g_{1,2}$ is a deformation retract of $V_1 \setminus C_1$, where C_1 denotes the union of all components of V_2^c contained in V_1, and $g_{1,2}$ is some graph in $(V_1 \setminus C_1) \setminus V_2$. It is enough to show that in every component C of $(V_1 \setminus C_1) \setminus V_2$ there is a graph g such that C contains $(\partial C \cap \partial V_2) \cup g$ as a deformation retract (by ∂ we mean the boundary). By a standard triangulation argument for C, the last assertion is true if some loop of ∂C is not contained in ∂V_2. Now, suppose $\partial C \subset \partial V_2$. This means that C is a component of V_2^c. Now, by the definition of C_1, we have $C \subset C_1$. But this is impossible, since, by the definition of C, $C \subset V_1 \setminus C_1$.

(D) Denote the fact proved in (C) by

$$V_1 \setminus C_1 \overset{\text{d.ret.}}{\to} V_2 \cup g_{1,2}.$$

Similarly, $V_2 \setminus C_2 \overset{\text{d.ret.}}{\to} V_3 \cup g_{2,3}$, where C_2 and $g_{2,3}$ are defined analogously, for all indices enlarged by one. Since C_2 is in the interior of V_2 (as an open set), it is in the first above deformation the image of itself only, whence

$$V_1 \setminus (C_1 \cup C_2) \overset{\text{d.ret.}}{\to} (V_2 \setminus C_2) \cup g_{1,2} \overset{\text{d.ret.}}{\to} V_3 \cup g_{1,3},$$

where $g_{1,3} \supset g_{1,2} \cup g_{2,3}$. By induction we obtain

$$V_1 \setminus (C_1 \cup C_2 \cup \ldots \cup C_n) \overset{\text{d.ret.}}{\to} V_{n+1} \cup g_{1,n+1}.$$

In particular, this proves that $V_1 \setminus \bigcup_{i=1}^{n} C_i$ is connected for each n and, the more, $X \setminus \bigcup_{i=1}^{n} C_i$ is connected.

Now, by (A) and by the fact that none of C_i is a disc (recall the definition of V_n in (B) and that of C_n in (C)), we see that, except for a finite number, the sets C_n are empty. Hence, applying the argument (C) to V_n (with n large enough) instead of V_1, we obtain

$$V_n \overset{\text{d.ret.}}{\to} V_{n+1} \cup g_{n,n+1}.$$

For fundamental groups this means that

$$\pi(V_n) = \pi(V_{n+1}) \times G_{n,n+1},$$

where $G_{n,n+1}$ is some free group and by $\times$ we denote the free product. By iteration, for all $m > n$,

$$\pi(V_n) = \pi(V_m) \times G_{n,n+1} \times G_{n+1,n+2} \times \ldots \times G_{m-1,m}.$$

Since $\pi(V_n) = \tau(\pi(U_n))$ is finitely generated, it follows that the groups $G_{m-1,m}$ are trivial, except for a finite number, and the lemma is proved.

LEMMA 2. *Let X be a compact connected two-manifold without boundary and Y some compact space of dimension one. If a continuous mapping ψ: $X \to Y$ is "onto" and monotonic (i.e., $\psi^{-1}(y)$ is connected for each $y \in Y$), then the homomorphism ψ_*: $\pi(X) \to \pi(Y)$ of fundamental groups induced by ψ is also "onto".*

Proof. Let γ be an arc in Y, and a, b its end points. We ought to show that there exists a homotopy class α of paths in X transformed by ψ into paths in Y homotopic to the path traversing γ. Fix some $a_0 \in \psi^{-1}(a)$ and $b_0 \in \psi^{-1}(b)$. Set $\Gamma = \psi^{-1}(\gamma)$. By monotonicity of ψ, Γ is a continuum in X, and hence Lemma 1 is valid. There exists a homotopy class α of paths from a_0 to b_0 in X having representants arbitrarily near to Γ. Thus, by uniform continuity of ψ, the paths of $\psi(\alpha)$ traverse arbitrarily near to γ. But in the one-dimensional space Y, homotopic paths traverse a common arc belonging to the same homotopy class. The arc is now equal to γ, because of the "arbitrarily near" argument.

Proof of Theorem 3. Suppose first that X is a manifold with boundary ∂X. Recall that, by the assumptions of Theorem 3, φ is a homeomorphism, whence ∂X is invariant. All the points of X are recurrent. Since ∂X is a one-manifold, we can apply Theorem 2 of [1], whence we obtain $\partial X \subset M$. Now, by transitivity of the system (X, φ) the assertion follows directly from Theorem 4.6 of [4]. The proper part of the proof dealing with manifolds without boundary is as follows: Let x_0 be a transitive point. If $x_0 \in M$, the assertion follows, if not, then repeating the construction made in the proof of Theorem 2 in [1], we can find a point $y \in X \setminus M$ with the closed orbit

$$Y = \mathrm{cl}\{\varphi^n(y),\ n = 1, 2, \ldots\}$$

nowhere dense in X. Let ψ be the element of S_φ for which $\psi(x_0) = y$. By the w.a.p. assumption, ψ is continuous and $\psi(X) = Y$. Observe that Y is of topological dimension one. By a general factorization theorem for continuous mapping (see, e.g., [6]) we have

$$\psi = \psi_2 \psi_1,$$

where $\psi_1\colon X \to \hat{Y}$, $\psi_2\colon \hat{Y} \to Y$, ψ_1 is continuous, "onto" and monotonic, ψ_2 is continuous, "onto" and zero-dimensional (i.e., $\psi_2^{-1}(y)$ is a zero-dimensional set for all $y \in Y$). Moreover, By the Hurewicz Theorem (cf., e.g., [2]), we have

$$1 = \dim Y = \dim \psi_2(\hat{Y}) \geqslant \dim \hat{Y} - 0,$$

whence $\dim \hat{Y} = 1$. On $\hat{Y}$ define $\hat{\varphi}$ by the formula $\hat{\varphi}\psi_1(x) = \psi_1 \varphi(x)$. An easy calculation (using monotonicity of ψ_1 and the commuting of φ and ψ) shows that the above definition is correct and that $(\hat{Y}, \hat{\varphi})$ is a w.a.p. transitive system. Moreover, $\psi_1(M) \subset \hat{M}$ and $\psi_2(\hat{M}) \subset M$, where $\hat{M}$ denotes the union of minimal sets for $(\hat{Y}, \hat{\varphi})$. By Lemma 2, the fundamental group of $\hat{Y}$ is finitely generated. Thus $\hat{Y}$ contains no more than finitely many loops. Since $\hat{\varphi}$ is also a homeomorphism, the union of the loops is a $\hat{\varphi}$-invariant finite graph $\hat{G}$. By Theorem 2 in [1] we have $\hat{G} \subset \hat{M}$. Write $\hat{y} = \psi_1(x_0)$. Since $\psi_2(\hat{y}) = y \notin M$, we have $\hat{y} \notin \hat{M}$. Now, there is exactly one arc $\hat{\gamma}$ in $\hat{Y}$ joining $\hat{y}$ with $\hat{M}$ (otherwise, we would have a loop not contained in $\hat{M}$). The arcs $\hat{\gamma}$ and $\hat{\varphi}(\hat{\gamma})$ may (a) have a common point p out of $\hat{M}$ or (b) be disjoint out of $\hat{M}$.

For (a) assume that p is the earliest such point. The only arc joining $\hat{y}$ with $\hat{\varphi}(\hat{y})$ contains p. By an easy induction we infer that the only arc joining $\hat{y}$ with $\hat{\varphi}^n(\hat{y})$ contains all the points $p, \hat{\varphi}(p), \ldots, \hat{\varphi}^{n-1}(p)$.

For (b) recall (the Lemma in the Introduction) that the sets of recurrent points for $\hat{\varphi}$ and $\hat{\varphi}^n$ are equal. Also the unions of minimal sets for these mappings have been shown in Theorem 2.11 in [4] to be the same. Thus we may assume that all the images $\hat{\varphi}^n(\hat{\gamma})$ are pairwise disjoint out of $\hat{M}$; otherwise, for some $\hat{\varphi}^n$ we would have case (a).

Now, since x_0 is recurrent, there exists an arc in X containing some sequence $\varphi^{n_k}(x_0)$ convergent to x_0. The arc may be represented as a union of arcs γ_k decreasing to x_0, such that γ_k contains x_0 and all $\varphi^{n_j}(x_0)$ for $j \geqslant k$. The images $\psi_1(\gamma_k)$ are path-connected subsets of Y, containing $\hat{y}$ and $\hat{\varphi}^{n_j}(\hat{y})$, respectively. Thus, in case (a) each of them contains the whole closed orbit of p, in particular the point $\hat{\varepsilon}(p)$ of $\hat{M}$, and in case (b), directly the points of $\hat{M}$. But, by the continuity of ψ_1, $\psi_1(\gamma_k) \to \hat{y}$, and, since $\hat{M}$ is closed, $\hat{y} \in \hat{M}$. Thus we have obtained a contradiction and we are done.

The theorem proved leads us to the following corollary, which is now a direct application of the Montgomery–Sine–Thomas classification [5]:

COROLLARY. *The only w.a.p. transitive system on a compact two-manifold is the minimal torus rotation.*

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