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ON INFINITE PRODUCTS OF INDEPENDENT RANDOM ELEMENTS ON METRIC SEMIGROUPS

BY

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It is well known that, for sums of independent real random variables, convergence in distribution, in probability, and with probability one are equivalent. This result was generalized in the recent years by several authors (Loynes [7], Csiszár [3], Tortrat [11], [12], Galmarino [4] and others) to infinite products of random elements with values in topological groups.

The aim of this paper is to establish a generalization of these results to infinite products of random elements taking values in a metric semigroup which satisfies conditions (L) and (R). Section 1 is preliminary. In Section 2 we consider families of accumulation points associated with infinite products of elements in a metric semigroup and we give algebraic characterizations of convergence of such products in terms of these families. In our investigations these characterizations play a role similar to those obtained by means of "tail idempotent" in [3].

In Section 3 we find a characterization of convergence in probability for infinite products of independent random elements with values in a metric semigroup in terms of the support of accumulation point families associated with these products (Theorem 3.1). As an application we derive conditions analogous to those in Theorem 1 of [1].

In Section 4 we prove a sufficient condition for the infinite product of random elements to converge with probability one (Theorem 4.1). From this result it follows immediately that, for infinite products of independent random elements with values in an Abelian semigroup, convergence in probability and with probability one are equivalent. Theorem 1 of Loynes [7] follows also from this result.

Results of this paper were partially announced in [2].

1. Preliminaries. Let S be a metric space. The Borel σ-algebra of S will be denoted by $\mathscr{B}$. By a *probability measure* on S we mean a non-negative, σ-additive tight measure μ on $\mathscr{B}$, i.e., a measure satisfying

$$\mu(A) = \sup_{K \subseteq A} \mu(K) \quad (K \text{ compact})$$

for every $A \in \mathscr{B}$, and such that $\mu(S) = 1$. The set of all probability measures on S will be denoted by $\tilde{S}$.

Now, let us consider the family of sets of the form

$$V_\mu(f_1, \ldots, f_k;\ \varepsilon_1, \ldots, \varepsilon_k) = \Big\{\nu \in \tilde{S};\ \Big|\int f_i d\nu - \int f_i d\mu\Big| < \varepsilon_i,\ i = 1, 2, \ldots, k\Big\},$$

where $f_1, f_2, \ldots, f_k$ are bounded real-valued continuous functions on S, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ are positive numbers. It is easy to verify that the family of sets obtained by varying $k, f_1, f_2, \ldots, f_k, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ satisfies the axioms of a basis for a topology. We shall refer to it as the weak topology in $\tilde{S}$, and $\tilde{S}$ will always be considered as a topological space with that topology. It is known that $\tilde{S}$ is metrizable (see [14]). We say that $\Pi \subseteq \tilde{S}$ is *uniformly tight* if for every real positive number ε there exists a compact set K such that $\mu(K) > 1 - \varepsilon$ for every $\mu \in \Pi$. Prokhorov's Theorem asserts that every uniformly tight family $\Pi \subseteq \tilde{S}$ is conditionally compact. On the other hand, if μ_n converges weakly to μ, and $\mu_n, \mu \in \tilde{S}$, then the family $\{\mu, \mu_1, \ldots, \mu_n, \ldots\}$ is uniformly tight (see [6]). So $\Pi \subseteq \tilde{S}$ is conditionally compact iff each sequence (μ_n), $\mu_n \in \Pi$, contains a subsequence $(\mu_{n'})$ such that $\{\mu_{n'}\}$ is uniformly tight.

We say that a mapping X from a probability space $(\Omega, \mathfrak{S}, \mathrm{P})$ into S is a *random element* if it is measurable (in the sense that $X^{-1}\mathscr{B} \subseteq \mathfrak{S}$).

Let $(X_n)_{n\geqslant 1}$ be a sequence of random elements. Let us assume that the distributions μ_n of X_n belong to $\tilde{S}$. We say that the sequence $(X_n)_{n\geqslant 1}$ *converges in distribution to the random element* X if μ_n converge weakly to the distribution μ of X.

Let d be a fixed metric in S and let $(X_n)_{n=0}^{\infty}$ be a sequence of random elements defined on a common probability space $(\Omega, \mathfrak{S}, \mathrm{P})$. Let us assume that $\mu_n \in \tilde{S}$ $(n = 0, 1, \ldots)$, where μ_n is the distribution of X_n, and that $(\Omega, \mathfrak{S}, \mathrm{P})$ is a complete measure space. Then, for every n, $d(X_n, X_0)$ is a random variable (see [10], p. 9). We say that X_n *converges in probability to* X_0 if

$$\lim_n \mathrm{P}\{d(X_n, X_0) > \varepsilon\} = 0$$

for each positive real ε. If $\{\mu_n\}$ is conditionally compact, then the Cauchy criterion,

$$\lim_{m,n} \mathrm{P}\{d(X_n, X_m) > \varepsilon\} = 0 \quad \text{for each } \varepsilon > 0,$$

is necessary and sufficient for the stochastic convergence of the sequence X_n to some random element X (this is a slight alteration of I.1 in [13]).

It is well known that if a sequence X_n of random elements converges with probability one (i.e., for P-almost all $\omega \in \Omega$) to a random element X, then it converges also to X in probability. Also, if a sequence of random elements converges in probability to some random element X, then it converges to X in distribution.

Now, let S be a *metric semigroup*, i.e., a metric space together with an associative, jointly continuous binary operation. By $\mathscr{B} \times \mathscr{B}$ we shall denote the Cartesian product of the Borel σ-algebras on S, and by $\mathscr{A}$ the Borel σ-algebra on $S \times S$. Let $\mu, \nu \in \tilde{S}$. The completion of $\mathscr{B} \times \mathscr{B}$ with respect to $\mu \times \nu$ will be denoted by $(\mathscr{B} \times \mathscr{B})^{\mu \times \nu}$. It is known that

$$\mathscr{B} \times \mathscr{B} \subseteq \mathscr{A} \subseteq (\mathscr{B} \times \mathscr{B})^{\mu \times \nu}$$

(see [12], p. 281). Hence the mapping $(x, y) \to xy$ from $S \times S$ into S is measurable with respect to the σ-algebras $\mathscr{B}$ and $(\mathscr{B} \times \mathscr{B})^{\mu \times \nu}$. If $B \in \mathscr{B}$, then the sets

$$x^{-1}B = \{y; xy \in B\} \qquad [By^{-1} = \{x; xy \in B\}]$$

are Borel measurable and the mappings

$$x \to \nu(x^{-1}B) \qquad [y \to \mu(By^{-1})]$$

are $\mathscr{B}^{\mu}$ $[\mathscr{B}^{\nu}]$ measurable, provided $\mu, \nu \in \tilde{S}$ ($\mathscr{B}^{\mu}$ and $\mathscr{B}^{\nu}$ denote the completion of $\mathscr{B}$ with respect to μ and ν, respectively). Thus we can define the convolution $\mu\nu$ of two probability measures by the formula

$$\mu\nu(B) = \mu \times \nu\{(x, y); xy \in B\} = \int_S \nu(x^{-1}B)\mu(dx) = \int_S \mu(By^{-1})\nu(dy) \tag{1.1}$$

for $B \in \mathscr{B}$. It is known that $\mu\nu \in \tilde{S}$ and that the convolution is associative and jointly continuous. Thus $\tilde{S}$ with the weak topology and the operation of convolution is a metric semigroup. (For the proofs, see [11] and [12].) By the *support of* μ we shall mean the set of all $x \in S$ having the property that $\mu(U) > 0$ for each open neighbourhood U of x. The support of μ will be denoted by $C(\mu)$. It is well known that

$$C(\mu\nu) = \overline{C(\mu)C(\nu)}. \tag{1.2}$$

If $\Pi \subseteq \tilde{S}$, then by the *support* $C(\Pi)$ *of* Π we shall mean the set

$$C(\Pi) = \overline{\bigcup_{\mu \in \Pi} C(\mu)}.$$

If X and Y are two random elements defined on a complete probability space and having tight distributions, then their product $Z = XY$ is also a random element (see [10], p. 9). Moreover, if X and Y are independent and μ, ν are their distributions, then the composition XY has the distribution $\mu\nu$ (provided μ and ν are tight and the basic probability space is complete).

Now, let S be a topological semigroup. We say that S *satisfies condition* (R) [(L)] if, for all compact subsets A and B of S,

$$A^{-1}B = \bigcup_{a\in A} a^{-1}B \qquad [AB^{-1} = \bigcup_{b\in B} Ab^{-1}]$$

is compact.

The semigroups having these properties have been introduced by Pym [9]. It is known that topological groups and compact semigroups satisfy both conditions (L) and (R).

LEMMA 1.1. *Let S be a metric semigroup satisfying* (R) [(L)]. *Let us suppose that $A^{-1}B \subseteq U$ $[AB^{-1} \subseteq U]$ for compact subsets $A, B \subseteq S$ and an open set U. Then there are open subsets V and W such that $A \subseteq V$, $B \subseteq W$ and $V^{-1}W \subseteq U$ $[VW^{-1} \subseteq U]$.*

Proof. Let us suppose that $a \in A$ and $b \in B$. Let

$$V_n = \{x;\, d(x, a) < 1/n\} \quad \text{and} \quad W_n = \{x;\, d(x, b) < 1/n\}.$$

Next, let

$$V_n(S\setminus U)\cap W_n \neq \emptyset \quad \text{for every } n.$$

Then $v_n s_n = w_n$ for some $v_n \in V_n$, $w_n \in W_n$, $s_n \in S\setminus U$. Let

$$D = \{a, v_1, v_2, \ldots\} \quad \text{and} \quad F = \{b, w_1, w_2, \ldots\}.$$

Then D and F are compact, and $s_n \in v_n^{-1}w_n \subseteq D^{-1}F$. Since $D^{-1}F$ is compact, there is a subsequence $s_{n'}$ such that $s_{n'}$ converges to $s_0 \in S\setminus U$. Therefore, there exists a positive integer N such that

$$V_n(S\setminus U)\cap W_n = \emptyset \quad \text{for } n \geqslant N,$$

which means that $V_n^{-1}W_n \subseteq U$. The remaining part of the proof follows immediately from the compactness of A and B.

From this lemma it is easy to obtain

COROLLARY 1.1. *Let S be a metric semigroup satisfying* (R). *Let $s \in S$ and let U be an open neighbourhood of s. Then there exists an open neighbourhood V of s such that*

$$V(s^{-1}V) \subseteq U.$$

Now, let $\mu, \nu \in \tilde{S}$ and $A, B, A^{-1}B, AB^{-1} \in \mathscr{B}$. It is easy to check the following formulas:

$$\mu(AB^{-1}) \geqslant \mu\nu(A) + \nu(B) - 1, \tag{1.3}$$

$$\nu(A^{-1}B) \geqslant \mu\nu(B) + \mu(A) - 1. \tag{1.4}$$

We have the following simple but very useful

LEMMA 1.2. *Let S be a metric semigroup satisfying* (L) [(R)]. *Then $\tilde{S}$ also satisfies* (L) [(R)].

The lemma follows directly from the characterization of weak compactness and from inequality (1.3) [(1.4)].

2. Infinite products in metric semigroups. In this section we prove some algebraic characterizations of convergence of infinite products in metric semigroups which are needed in the following sections.

Throughout this section, S denotes a metric semigroup satisfying condition (R).

Let $(x_n)_{n\geqslant 1}$ be a sequence of elements from S. For the sake of brevity, throughout this section we use the following notation:

$$y_m^n = x_m x_{m+1} \ldots x_n \ (m \leqslant n), \qquad y_n = y_1^n.$$

Let us assume that $\{y_n\}_{n=1}^{\infty}$ is conditionally compact. By N_k we shall denote the set of all accumulation points of the sequence $(y_k^n)_{n\geqslant k}$ $(k = 1, 2, \ldots)$, and by N_∞ the set of all accumulation points of all sequences $(\gamma_k)_{k\geqslant 1}$, where $\gamma_k \in N_k$. Finally, by Λ we shall denote the set of all accumulation points of the double sequence $(y_m^n)_{m\leqslant n}$.

First we make the following simple observation about N_k, N_∞ and Λ.

LEMMA 2.1. *Suppose that the set $\{y_n\}_{n=1}^{\infty}$ is conditionally compact. Then $\{y_k^n\}_{n=k}^{\infty}$ is conditionally compact for each $k = 1, 2, \ldots$, and $\{y_m^n\}_{m\leqslant n}$ $(m, n = 1, 2, \ldots)$ is conditionally compact. Moreover,*

(i) *for every $\gamma_k \in N_k$ there exists $\gamma \in N_\infty$ such that $\gamma_k\gamma = \gamma_k$ $(k = 1, 2, \ldots)$;*

(ii) $\Lambda \subseteq N_k^{-1} N_k$ $(k = 1, 2, \ldots)$;

(iii) $N_\infty \subseteq \Lambda \subseteq N_\infty^{-1} N_\infty \cap N_\infty N_\infty^{-1}$.

Proof. It follows from the assumption that $A_1 = \overline{\{y_1, y_2, \ldots\}}$ is compact. Since $y_n = y_{k-1} y_k^n$, we have

$$y_k^n \in y_{k-1}^{-1} y_n \subseteq A_1^{-1} A_1 \quad \text{for } k \leqslant n \ (k, n = 1, 2, \ldots),$$

and so the compactness of $\{y_k^n\}_{n=k}^{\infty}$ and $\{y_m^n\}_{m\leqslant n}$ follows immediately from the compactness of $A_1^{-1}A_1$ (let us recall that S satisfies (R)).

Properties (i) and (ii) follow easily from the formula

$$y_k^l y_{l+1}^m = y_k^m \quad (k \leqslant l < m), \tag{2.1}$$

the continuity of multiplication, and the definition of Λ and N_k.

The inclusion $\Lambda \subseteq N_\infty^{-1} N_\infty$ follows from (ii). The inclusion $\Lambda \subseteq N_\infty N_\infty^{-1}$ can be derived, in a similar way, from (2.1); and Tortrat proved in [13] that $N_\infty \subseteq \Lambda$.

LEMMA 2.2. *Let us assume that* $\{y_n\}_{n\geqslant 1}$ *is conditionally compact. The following conditions are equivalent*:

(i) $$(y_n)_{n\geqslant 1} \text{ is convergent},$$

(ii) $$\gamma\Lambda = \gamma \quad \text{for a } \gamma \in N_1,$$

(iii) $$\gamma N_\infty = \gamma \quad \text{for a } \gamma \in N_1.$$

Proof. It is easy to observe that condition (ii) is equivalent to the following one:

(ii′) $$\lim_{\substack{m\leqslant n\\ m,n\to\infty}} \gamma y_m^n = \gamma \quad \text{for a } \gamma \in N_1.$$

It is also obvious that (ii) $\Rightarrow$ (iii). So it remains to prove that (i) $\Rightarrow$ (ii), (ii′) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii).

The implication (i) $\Rightarrow$ (ii) is easy: if $y_{n'}^{m'} \to \lambda$ for a double sequence (m', n'), $n' \leqslant m'$, then by the convergence of y_n to γ ($N_1 = \{\gamma\}$ by (i)) and by (2.1) we have $\gamma\lambda = \gamma$.

Now, let

$$\lim_{\substack{m\leqslant n\\ m,n\to\infty}} \gamma y_m^n = \gamma \quad \text{for a } \gamma \in N_1.$$

By the definition of N_1 there is a subsequence n' such that $\gamma = \lim y_{n'}$. Let U be an open neighbourhood of γ. By Corollary 1.1 there exists an open neighbourhood V of γ such that

$$V(\gamma^{-1}V) \subseteq U.$$

If n_0 is a positive integer in $\{n'\}$ such that $y_{n'} \in V$ and $\gamma y_m^n \in V$ for $n', n, m \geqslant n_0$, then

$$y_n = y_{n_0} y_{n_0+1}^n \in V(\gamma^{-1}V) \subseteq U \quad \text{for } n > n_0.$$

Hence $\lim y_n = \gamma$, and (ii′) $\Rightarrow$ (i).

It remains to prove that (iii) $\Rightarrow$ (ii). Let us suppose that, for an element $\gamma \in N_1$, $\gamma N_\infty = \gamma$. Let $\delta \in N_\infty^{-1} N_\infty$, i.e., $\gamma_1 \delta = \gamma_2$ for some $\gamma_1, \gamma_2 \in N_\infty$. Then

$$\gamma\delta = \gamma\gamma_1\delta = \gamma\gamma_2 = \gamma.$$

Since $\Lambda \subseteq N_\infty^{-1} N_\infty$, we have (ii), which completes the proof.

Now let $(x_n)_{n\geqslant 1}$ be a sequence of elements in S. If y_k^n is convergent, whenever n tends to infinity, for every k, then we say that (x_n) is *compositionally convergent*. In such a case we shall denote $\lim\limits_{n\to\infty} y_k^n$ by $\tilde{y}_k$.

LEMMA 2.3. *Let us suppose that* $\{y_n\}$ *is conditionally compact. The following conditions are equivalent*:

(i) $(x_n)_{n\geqslant 1}$ *is compositionally convergent,*

(ii) $\gamma\Lambda = \gamma$ *for every* $\gamma \in N_\infty$,

(iii) N_∞ *is a left-zero semigroup, i.e.,* $\gamma_1\gamma_2 = \gamma_1$ *for every* $\gamma_1, \gamma_2 \in N_\infty$.

Proof. It follows from Lemma 2.2 that (i) $\Rightarrow$ (ii). It is also obvious that (ii) $\Rightarrow$ (iii). The proof that (iii) $\Rightarrow$ (ii) is quite similar to that of the implication (iii) $\Rightarrow$ (ii) in Lemma 2.2. Thus it remains to prove that (ii) $\Rightarrow$ (i). Let us assume that (ii) is valid. If $\gamma_k \in N_k$, then by Lemma 2.1 (i) there exists an element $\gamma \in N_\infty$ such that $\gamma_k\gamma = \gamma_k$. Since $\gamma\Lambda = \gamma$ (by (ii)), we have

$$\gamma_k\Lambda = \gamma_k\gamma\Lambda = \gamma_k\gamma = \gamma_k.$$

Thus, by Lemma 2.2 (ii), $(x_n)_{n\geqslant 1}$ is compositionally convergent.

3. Convergence in probability. In the sequel, S will denote a metric semigroup satisfying condition (R) and having a *left-subinvariant metric d* (i.e., such that $d(zx, zy) \leqslant d(x, y)$ for $x, y, z \in S$). It is well known that metric groups have such a metric (in fact, a left-invariant one, see [5]). Michael has proved in [8] that locally compact semigroups satisfying the second countability axiom and condition (L) also have such a metric.

Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random elements (i.r.e.) with values in S. By μ_n we shall denote the distribution of X_n. We always assume that all random elements considered in this paper have tight distributions and that the probability space $(\Omega, \mathfrak{S}, \mathrm{P})$ on which they are defined is complete. For the sake of brevity, we shall use in the sequel the following notation:

$$Y_m^n = X_m X_{m+1} \ldots X_n \ (m \leqslant n), \qquad Y_1^n = Y_n.$$

Distributions of X_n, Y_m^n and Y_n will be denoted by μ_n, ν_m^n and ν_n, respectively. Clearly,

$$\nu_m^n = \mu_m\mu_{m+1} \ldots \mu_n, \qquad \nu_n = \mu_1\mu_2 \ldots \mu_n.$$

Since $\tilde{S}$ (the semigroup of all probability measures on S) is a metric semigroup satisfying (R), we can apply the results of the preceding section to this semigroup. When dealing with this semigroup, we shall write μ and ν instead of x and y, respectively. Let N_k, N_∞ and Λ denote the sets defined as in the previous section for the sequence $(x_n)_{n\geqslant 1} = (\mu_n)_{n\geqslant 1}$.

Now we state and prove the main result of this section.

THEOREM 3.1. *Let* $(X_n)_{n\geqslant 1}$ *be a sequence of independent random elements taking values in* S. *Let us assume that* $\{\nu_n\}_{n\geqslant 1}$ *is conditionally compact. Then the following statements are equivalent*:

(i) Y_n *converges in probability,*

(ii) $xC(\Lambda) = x$ *for every* $x \in C(N_1)$,

(iii) $xC(N_\infty) = x$ *for every* $x \in C(N_1)$.

Proof. Suppose that (i) holds, i.e., Y_n converges in probability to a random element $\tilde{Y}_1$. Let $\tilde{\nu}_1$ be the distribution of $\tilde{Y}_1$. Let λ be an arbitrary element of Λ and let $e \in C(\lambda)$. Thus there is a subsequence $(\nu_{n_{2k-1}}^{n_{2k}})_{k\geqslant 1}$ $(n_k < n_{k+1},\ k = 1, 2, \ldots)$ of (ν_m^n) such that

$$\lim \nu_{n_{2k-1}}^{n_{2k}} = \lambda. \tag{3.1}$$

Let us write

$$Z_{2k-1} = Y_{n_{2k-1}-1}, \quad Z_{2k} = Y_{n_{2k}} \quad (k = 1, 2, \ldots)$$

and

$$T_k = X_{n_{2k-1}} \ldots X_{n_{2k}}.$$

Let ϱ_k denote the distribution of T_k. Without loss of generality we may assume that Z_k converges to $\tilde{Y}_1$ with probability one. By the construction of Z_k and T_k we have

$$Z_{2k} = Z_{2k-1} T_k \quad (k = 1, 2, \ldots). \tag{3.2}$$

Let $\{U_n\}_{n\geqslant 1}$ be a basis of open neighbourhoods at e. By (3.1) we have

$$\liminf \varrho_k(U_m) \geqslant \lambda(U_m) > 0 \quad (m = 1, 2, \ldots).$$

Then, from the Borel-Cantelli lemmas it follows that

$$\mathrm{P}(\limsup_k \{T_k \in U_m\}) = 1 \quad (m = 1, 2, \ldots).$$

Let us write

$$E = \bigcap_{m\geqslant 1} \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} \{T_k \in U_m\}$$

and let $D \subseteq E$ be such that $P(D) = 1$ and $Z_k(\omega) \to \tilde{Y}_1(\omega)$ whenever $\omega \in D$.

Now, for every $\omega_0 \in D$ we choose a subsequence $T_{k'}$ such that $T_{k'}(\omega_0)$ converges to e if $k' \to \infty$. On the other hand,

$$Z_{k'}(\omega_0) \to \tilde{Y}_1(\omega_0) \quad \text{if } k' \to \infty.$$

By (3.2), we have

$$\tilde{Y}_1 = \tilde{Y}_1 e \ (\mathrm{mod}\ \mathrm{P}).$$

In order to conclude that (ii) holds it remains to show that $x = xe$ for each $x \in C(\tilde{\nu}_1)$. Suppose that there is an element $x \in C(\tilde{\nu}_1)$ such that $xe \neq x$. So there are open sets U, V and W such that

$$x \in U, \quad x \in W, \quad U \cap V = \emptyset, \quad We \subseteq V.$$

Hence

$$0 < \mathrm{P}\{\tilde{Y}_1 \in U \cap W\} \leqslant \mathrm{P}\{\tilde{Y}_1 \in U;\ \tilde{Y}_1 e \in V\} = \mathrm{P}\{\tilde{Y}_1 \in U \cap V\} = 0,$$

a contradiction. Thus we have $xC(\lambda) = x$ for every $x \in C(\tilde{\nu}_1)$ and every $\lambda \in \Lambda$. Since $x^{-1}x$ is closed, we have (ii).

Since $N_\infty \subseteq \Lambda$, (ii) $\Rightarrow$ (iii). Our proof will be completed if we show that (iii) $\Rightarrow$ (i).

Let us write $H = C(N_\infty)$. Suppose that (iii) holds. Since all probability measures with finite support are dense in $\tilde{S}$, we obtain $\gamma N_\infty = \gamma$ for every $\gamma \in N_1$. From Lemma 2.2 it follows that ν_n converges to a measure $\tilde{\nu}_1$ and that $N_1 = \{\tilde{\nu}_1\}$. Given an arbitrary $\varepsilon > 0$, set

$$U = \bigcup_{e,f\in H} \{s \in S;\, d(se, f) < \varepsilon\},$$

$$V = \bigcup_{e\in H} \{s \in S;\, d(s, e) < \varepsilon/2\}.$$

Since d is left-subinvariant, we have

$$VV^{-1} \subseteq U. \tag{3.3}$$

Using the left-subinvariance of d, the continuity of convolution and inequality (1.3) we obtain

$$\begin{aligned} \mathrm{P}\Big(\bigcup_{e,f\in H} \{d(Y_n e, Y_m f) < \varepsilon\}\Big) & \\ \geqslant \mathrm{P}\Big(\bigcup_{e,f\in H} \{d(Y^n_{m+1} e, f) < \varepsilon\}\Big) &= \nu^n_{m+1}(U) \\ \geqslant \nu^n_{m+1}(VV^{-1}) &\geqslant \gamma_{m+1}(V) + \gamma_{n+1}(V) - 1 \end{aligned} \tag{3.4}$$

for $n > m$ and a sequence $(\gamma_k)_{k\geqslant 1}$ $(\gamma_k \in N_k)$. Since V is an open neighbourhood of H, we have

$$\lim_k \gamma_k(V) = 1.$$

Thus, by (3.4),

$$\lim_{\substack{m,n\to\infty \\ m\leqslant n}} \mathrm{P}\Big(\bigcup_{e,f\in H} \{d(Y_n e, Y_m f) < \varepsilon\}\Big) = 1. \tag{3.5}$$

Put

$$F = \bigcup_{e\in H} \{s \in S;\, d(se, s) > \varepsilon\}.$$

We have

$$\bar{F} \subseteq S \setminus C(\tilde{\nu}_1), \tag{3.6}$$

and so

$$\limsup \nu_n(F) \leqslant \limsup \nu_n(\bar{F}) \leqslant \tilde{\nu}_1(\bar{F}) = 0.$$

Thus

$$\lim_n \nu_n(F) = 0$$

which means that

$$\lim_n \mathrm{P}\Big(\bigcap_{e\in H} \{d(Y_n e, Y_n) \leqslant \varepsilon\}\Big) = 1. \tag{3.7}$$

Finally, by the inequality (easy to verify)

$$\mathrm{P}\big(\{d(Y_n, Y_m) \geqslant 3\varepsilon\}\big) \leqslant \mathrm{P}\big(\bigcup_{e\in H}\{d(Y_n, Y_n e) > \varepsilon\}\big) + \\ + \mathrm{P}\big(\bigcap_{e,f\in H}\{d(Y_n e, Y_m f) \geqslant \varepsilon\}\big) + \mathrm{P}\big(\bigcup_{f\in H}\{d(Y_m f, Y_m) > \varepsilon\}\big)$$

and by (3.5) and (3.7), we obtain

$$\lim_{n,m\to\infty} \mathrm{P}\big(\{d(Y_n, Y_m) \geqslant 3\varepsilon\}\big) = 0.$$

Consequently, Y_n satisfies the Cauchy criterion, and hence there is a random element $\tilde{Y}_1$ such that Y_n converges in probability to $\tilde{Y}_1$. This completes the proof.

We say that the sequence $(X_n)_{n\geqslant 1}$ of independent random elements is *compositionally convergent* [*in distribution, in probability, with probability one*] if, for every m, Y_m^n is convergent whenever n tends to infinity [in distribution, in probability, with probability one].

COROLLARY 3.1. *Let X_n be a sequence of i.r.e. Suppose that $\{\nu_n\}$ is conditionally compact. Then the following conditions are equivalent:*

(i) *$(X_n)_{n\geqslant 1}$ is compositionally convergent in probability,*

(ii) *$eC(\Lambda) = e$ for every $e \in C(N_\infty)$,*

(iii) *$C(N_\infty)$ is a left-zero semigroup.*

Proof. Suppose first that $(X_n)_{n\geqslant 1}$ is compositionally convergent in probability. Denote by C the support of the family $\{\tilde{\nu}_k\}_{k\geqslant 1}$, where

$$\tilde{\nu}_k = \lim_{n\to\infty} \nu_k^n.$$

By Theorem 3.1, $xC(N_\infty) = x$ for every $x \in C(\tilde{\nu}_k)$ $(k = 1, 2, \ldots)$, and so for every $x \in C$. Now, if μ_n converges weakly to μ, then by the definition of the support of measure and by the inequality

$$\liminf \mu_n(V) \geqslant \mu(V)$$

for every open V, we infer that

$$C(\mu) \subseteq \bigcup C(\mu_n).$$

Hence $C(N_\infty) \subseteq C$, and so $eC(N_\infty) = e$ for every $e \in C(N_\infty)$, which means that $C(N_\infty)$ is a left-zero semigroup. This proves that (i) $\Rightarrow$ (iii).

Now, to show that (iii) $\Rightarrow$ (ii), let $C(N_\infty)$ be a left-zero semigroup. Hence N_∞ is also a left-zero semigroup (since the probability measures with finite support are dense in $\tilde{S}$ and the convolution is continuous). From Lemma 2.2 it follows that $\gamma\lambda = \gamma$ for every $\gamma \in N_\infty$ and every $\lambda \in \Lambda$. Hence

$$C(\gamma)\,C(\lambda) \subseteq C(\gamma) \quad \text{for every } \gamma \in N_\infty,\ \lambda \in \Lambda.$$

By this inclusion we have

$$eC(\lambda) = eC(\gamma)C(\lambda) \subseteq eC(\gamma) = e \quad \text{for every } e \in C(N_\infty),\ \lambda \in \Lambda.$$

So we have proved that (iii) $\Rightarrow$ (ii).

Finally, suppose that (ii) holds. Since $N_\infty \subseteq \Lambda$ (Lemma 2.1), $C(N_\infty) \subseteq C(\Lambda)$. From (ii) it follows that, in particular, $eC(N_\infty) = e$ for every $e \in C(N_\infty)$ which means that $C(N_\infty)$ is a left-zero semigroup. Since the probability measures with finite support are dense in $\tilde{S}$, N_∞ is also a left-zero semigroup. Thus, by Lemma 2.3, $(X_n)_{n\geqslant 1}$ is compositionally convergent (in distribution) and $\tilde{\nu}_k\gamma = \tilde{\nu}_k$ for every $\gamma \in N_\infty$ $(k = 1, 2, \ldots)$, where

$$\tilde{\nu}_k = \lim_{n\to\infty} \nu_k^n.$$

Since $\gamma e = \gamma$ for every $e \in C(N_\infty)$ and $\gamma \in N_\infty$, we have $\tilde{\nu}_k e = \tilde{\nu}_k$. Since $e^2 = e$, we obtain

$$C(\tilde{\nu}_k)e = C(\tilde{\nu}_k) \quad (k = 1, 2, \ldots),\ e \in C(\Lambda).$$

Let $x \in C(\tilde{\nu}_k)$. By this equality there is $y \in C(\tilde{\nu}_k)$ such that $ye = x$. Hence $xe = yee = ye = x$ and

$$xC(N_\infty) = x \quad \text{for every } x \in C(\tilde{\nu}_k)\ (k = 1, 2, \ldots).$$

By Theorem 3.1, $(X_n)_{n\geqslant 1}$ is compositionally convergent in probability.

4. Convergence with probability one. Throughout this section we assume that S is a metric semigroup satisfying (R) and (L) and having a left-subinvariant metric d.

The main result of this section is the following

THEOREM 4.1. *Let $(X_n)_{n\geqslant 1}$ be a sequence of i.r.e. taking values in S. Suppose that $\{\nu_n\}$ is conditionally compact. If*

$$x[C(N_\infty)C(\Lambda)^{-1}] = x \quad \textit{for every } x \in C(N_1), \tag{4.1}$$

then Y_n converges with probability one.

Proof. From Lemma 2.1 it follows that $\Lambda \subseteq N_\infty \Lambda^{-1}$. Hence

$$\bigcup_{\lambda\in\Lambda} C(\lambda) \subseteq C(N_\infty)C(\Lambda)^{-1}.$$

By (4.1), we have

$$\bigcup_{\lambda\in\Lambda} C(\lambda) \subseteq x^{-1}x \quad \text{for } x \in C(N_1).$$

Since $x^{-1}x$ is closed, $C(\Lambda) \subseteq x^{-1}x$, which gives

$$xC(\Lambda) = x \quad \text{for every } x \in C(N_1). \tag{4.2}$$

By Theorem 3.1, Y_n converges in probability to a random element $\tilde{Y}_1$ with the distribution $\tilde{\nu}_1$, and $N_1 = \{\tilde{\nu}_1\}$.

Let x be an arbitrary but fixed element of $C(\tilde{\nu}_1)$ and let W_r be the open ball with center x and radius $1/r$. From condition (R) it follows that $x^{-1}x$ is compact. By (4.2), $C(\Lambda)$ is compact, and so is $C(N_\infty)$. Thus $C(N_\infty)C(\Lambda)^{-1}$ is also compact. Let us observe that $x^{-1}W_r$ is an open neighbourhood of $C(N_\infty)C(\Lambda)^{-1}$. In virtue of Lemma 1.1, there are open neighbourhoods U_r and V_r of $C(N_\infty)$ and $C(\Lambda)$, respectively, such that

$$U_rV_r^{-1} \subseteq x^{-1}W_r \quad (r = 1, 2, \ldots). \tag{4.3}$$

Using the definition of N_∞ and the fact that U_r is an open neighbourhood of $C(N_\infty)$ we can choose, by induction, a sequence n_k ($n_k < n_{k+1}$, $k = 1, 2, \ldots$) such that

$$Y_{n_k} \to \tilde{Y}_1 \text{ with probability one,} \tag{4.4}$$

$$\mathrm{P}\{Y_{n_k+1}^{n_{k+1}} \in U_k\} \geqslant 1-2^{-k} \quad (k = 1; 2, \ldots). \tag{4.5}$$

The rest of the proof is very similar to the final part of the proof of Theorem 3.2 in Csiszár's paper [3].

We make use of a "lemma for events" asserting that if A_i and B_i ($i = 1, 2, \ldots, m$) are arbitrary random events such that, for each fixed i, A_i and B_i are independent, then

$$\mathrm{P}\Big(\bigcup_{i=1}^{m} A_i \cap B_i\Big) \geqslant \inf_{1\leqslant i\leqslant m} \mathrm{P}(B_i)\cdot \mathrm{P}\Big(\bigcup_{i=1}^{m} A_i\Big). \tag{4.6}$$

Applying (4.6) for the events

$$A_i = \{Y_{n_k+1}^{n_k+i} \notin x^{-1}W_r\} \quad \text{and} \quad B_i = \{Y_{n_k+i+1}^{n_{k+1}} \in V_r\},$$

where k, r are fixed positive integers and $i = 1, 2, \ldots, n_{k+1}-n_k-1$, and using (4.3), we obtain

$$\begin{aligned} \mathrm{P}\{Y_{n_k+1}^{n_{k+1}} \notin U_r\} &\geqslant \mathrm{P}\Big(\bigcup_{i=1}^{n_{k+1}-n_k-1} A_i \cap B_i\Big) \\ &\geqslant \inf_{n_k\leqslant n\leqslant n_{k+1}-1} \mathrm{P}\{Y_{n+1}^{n_{k+1}} \in V_r\}\cdot \mathrm{P}\Big(\bigcup_{n=n_k+1}^{n_{k+1}-1} \{Y_{n_k+1}^{n} \notin x^{-1}W_r\}\Big). \end{aligned} \tag{4.7}$$

Since V_r is an open neighbourhood of $C(\Lambda)$, we have

$$\inf_{n_k< n\leqslant n_{k+1}-1} \mathrm{P}\{Y_{n+1}^{n_{k+1}} \in V_r\} \geqslant \frac{1}{2} \quad \text{for } k \text{ large enough.} \tag{4.8}$$

Combining (4.5), (4.7) and (4.8) we obtain

$$\sum_{k=1}^{\infty} \mathrm{P}\Big(\bigcup_{n=n_k+1}^{n_{k+1}-1} \{Y_{n_k+1}^{n} \notin x^{-1}W_r\}\Big) < \infty.$$

Thus, by the Borel-Cantelli lemma, we have

$$Y^n_{n_k+1} \in x^{-1}W_r \quad (n = n_k+1, \ldots, n_{k+1}-1) \tag{4.9}$$

with probability one, except for a finite number of k's. Now, if O is an arbitrary open neighbourhood of x, then by Corollary 1.1 we obtain

$$W_r(x^{-1}W_r) \subseteq O \tag{4.10}$$

for sufficiently large r. Since x is an arbitrary element of $C(\tilde{\nu}_1)$, and $Y_n = Y_{n_k}Y^n_{n_k+1}$, we infer, using (4.4), (4.9) and (4.10), that Y_n converges with probability one.

Remark. The following simple example shows that condition (4.1) of Theorem 4.1 is not necessary.

Let $S = \{e, f\}$ be a right-zero semigroup, i.e., $ef = f = f^2, fe = e = e^2$, and let $X_n = e$. Then $Y_n = e$, but condition (4.1) is not satisfied.

Corollary 4.1. *Let $(X_n)_{n\geqslant 1}$ be a sequence of i.r.e. Suppose that $\{\nu_n\}$ is conditionally compact. If Y_n converges stochastically and*

$$C(\Lambda)C(\Lambda)^{-1} \subseteq C(\Lambda)^{-1}C(\Lambda),$$

then Y_n converges with probability one.

The corollary follows immediately from Theorems 3.1 and 4.1.

Remark. This corollary implies immediately that if $C(\Lambda)$ is in the center of S, i.e., if $\lambda s = s\lambda$ for every $s \in S$ and $\lambda \in C(\Lambda)$, then Y_n converges stochastically iff it converges with probability one. In particular, if S is Abelian, then convergence in probability and with probability one for infinite products of i.r.e. are equivalent.

Corollary 4.2. *Let $(X_n)_{n\geqslant 1}$ be a sequence of i.r.e. with values in S. Suppose that $\{\nu_n\}_{n\geqslant 1}$ is conditionally compact. If*

$$e[C(N_\infty)C(N_\infty)^{-1}] = e \quad \text{for every } e \in C(N_\infty), \tag{4.11}$$

then $(X_n)_{n\geqslant 1}$ is compositionally convergent with probability one.

Proof. Using the same arguments as in the first part of the proof of Theorem 4.1 we obtain

$$eC(N_\infty) = e \quad \text{for every } e \in C(N_\infty). \tag{4.12}$$

In particular, $C(N_\infty)$ is a left-zero semigroup. From Corollary 3.1 it follows that $(X_n)_{n\geqslant 1}$ is compositionally convergent in probability. By Theorem 3.1, for every $x \in C(N_k)$ $(k = 1, 2, \ldots)$ and for every $e \in C(N_\infty)$ we have

$$xe = x. \tag{4.13}$$

Now we prove that

$$N_\infty\Lambda^{-1} \subseteq N_\infty N_\infty^{-1}. \tag{4.14}$$

For if $\varrho \in N_\infty \Lambda^{-1}$, then there is $\lambda \in \Lambda$ such that $\varrho\lambda \in N_\infty$. By Lemma 2.1, there is $\gamma \in N_\infty$ such that $\lambda\gamma \in N_\infty$. Thus $\varrho\lambda\gamma \in N_\infty^2$. By Corollary 2.1, $N_\infty^2 = N_\infty$, whence $\varrho \in N_\infty N_\infty^{-1}$. By (4.11) and (4.14), we have

$$e[C(N_\infty)C(\Lambda)^{-1}] = e \quad \text{for every } e \in C(N_\infty). \tag{4.15}$$

From (4.13) and (4.15) it follows that

$$x[C(N_\infty)C(\Lambda)^{-1}] = x \quad \text{for every } x \in C(N_k).$$

By Theorem 4.1, $(X_n)_{n\geqslant 1}$ is compositionally convergent with probability one.

By Corollary 4.2 we obtain

COROLLARY 4.3. *Let $(X_n)_{n\geqslant 1}$ be a sequence of i.r.e. Suppose that $\{\nu_n\}_{n\geqslant 1}$ is conditionally compact. If $C(N_\infty)$ is a left-zero semigroup and*

$$C(N_\infty)C(N_\infty)^{-1} \subseteq C(N_\infty)^{-1}C(N_\infty),$$

then $(X_n)_{n\geqslant 1}$ is compositionally convergent with probability one.

Remarks. (i) If $C(N_\infty)$ is a left-zero semigroup and $C(N_\infty)$ is in the center of S, then Y_n converges with probability one. In particular, if each idempotent of S is in the center, then every sequence $(X_n)_{n\geqslant 1}$ of i.r.e. such that $C(N_\infty)$ is a left-zero semigroup is compositionally convergent with probability one. This statement has been proved in [1], Theorem 1 (iii) (with an additional assumption that S is finite and $C(N_\infty)$ is one-point). From Corollary 4.3 it follows also that, for infinite products of independent random elements with values in a metric group, the convergence in probability and with probability one are equivalent — the result obtained earlier by Loynes (Theorem 1 in [7]).

(ii) The assumption that the basic probability space $(\Omega, \mathfrak{S}, \mathrm{P})$ is complete is needed only to prove that every pair (X, Y) of random elements with values in S is measurable with respect to the Borel σ-algebra $\mathscr{A}$ in $S \times S$ and the σ-algebra $\mathfrak{S}$ in Ω (see [10], p. 9). However, this is always true if S is separable. Thus, in this case, the results of Sections 3 and 4 remain valid without any additional assumptions concerning the space $(\Omega, \mathfrak{S}, \mathrm{P})$.

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