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OPERATOR SEMI-STABLE PROBABILITY MEASURES ON BANACH SPACES

BY

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In this paper we define operator semi-stable probability measures on a real separable Banach space which are identified as limit laws. Further, we get a representation of the characteristic functionals of operator semi-stable probability measures.

1. Notation and preliminaries. Let X denote a real separable Banach space with norm $\|\cdot\|$ and with dual space X^*. By $\langle\cdot, \cdot\rangle$ we denote the dual pairing between X and X^*. Further, $B(X)$ will denote the algebra of continuous linear operators on X with norm topology. Given a subset F of $B(X)$, we denote by $\mathrm{Sem}(F)$ the closed multiplicative semigroup of operators spanned by F. The unit and zero operators will be denoted by I and 0, respectively.

A sequence $\{\mu_n\}$ of probability measures on X is said to converge to a probability measure μ on X if for every bounded continuous real-valued function f on X

$$\int_X f d\mu_n \to \int_X f d\mu.$$

The characteristic functional of μ is defined on X^* by

$$\mu(y) = \int_X e^{i\langle x,y\rangle}\mu(dx),$$

where $y \in X^*$. For an operator A from $B(X)$ and a probability measure μ on X let A denote the probability measure defined by $A\mu(E) = \mu(A^{-1}(E))$ for all Borel subsets E of X. It is easy to check the equations

$$A(\mu*\nu) = A\mu*A\nu, \qquad A\mu(y) = \hat{\mu}(A^*y),$$

where A^* denotes the adjoint operator. Moreover, $A_n\mu_n \to A\mu$ whenever $A_n \to A$ and $\mu_n \to \mu$. A probability measure μ on X is said to be *full* if its support is not contained in any proper hyperplane of X. By δ_x $(x \in X)$ we denote the probability measure concentrated at the point x.

A probability measure μ on X is said to be *infinitely divisible* whenever for every positive integer n there exists a probability measure μ_n such that $\mu = \mu_n^{*n}$, where the power is taken in the sense of convolution. Let μ be an infinitely divisible probability measure on X. Then for every $c \geqslant 0$ there exists an infinitely divisible measure ν on X such that $\hat{\nu}(y) = [\mu(y)]^c$. We denote ν by μ^c. The set $\{\mu^c\}_{c\geqslant 0}$ is an Abelian semigroup with the convolution as a semigroup operation, and the mapping $c \to \mu^c$ is a continuous homomorphism of the additive semigroup of non-negative real numbers onto $\{\mu^c\}_{c\geqslant 0}$ (Proposition 1.2 of [8]).

LEMMA 1. *Let μ and ν be probability measures on X and let $\{x_n\}$ be a sequence of elements of X such that*

$$\lim_{n\to\infty} \mu(y)\exp(i\langle x_n, y\rangle) = \hat{\nu}(y) \quad \textit{for all } y \in X^*.$$

Then there exists a unique element $x \in X$ such that $\mu = \delta_x * \nu$.

The lemma follows immediately from Lemma 1.1 of [8].

Given a probability measure μ on X, we define $\bar{\mu}$ by $\bar{\mu}(E) = \mu(-E)$, where $-E = \{-x\colon x \in E\}$. For any probability measure μ on X the measure $|\mu|^2 = \mu * \bar{\mu}$ is called the *symmetrization* of μ.

Let $[a]$ be the largest integer not greater than a.

2. Stating the problem. Let μ be a probability measure on X. We call μ *operator semi-stable* if its characteristic functional $\hat{\mu}$ satisfies the functional equation

$$[\hat{\mu}(y)]^c = \hat{\mu}(B^*y)e^{i\langle b, y\rangle} \quad \text{for all } y \in X^*, \tag{2.1}$$

where $B \in B(X)$, $b \in X$ and $c \in (0, 1)$. In the one-dimensional case, characteristic functionals which satisfy for all x an equation of the form

$$\varphi(x) = [\varphi(bx)]^a,$$

where $a > 0$ and $0 < b < 1$, have been considered by Lévy ([11], p. 204) and the solutions have been called by him *semi-stable*. Semi-stable measures on the real line have been studied by Kruglov in [9], by Pillai in [13] and by Rao and Ramachandran in [14]. Operator semi-stable measures on finite-dimensional spaces have been considered by Jajte in [6]. Kumar [10] has treated semi-stable measures on Hilbert spaces and proved that they are limit laws. We obtained a representation of the characteristic functionals of these laws in the same manner as Jajte did in [5] for stable probability measures.

PROPOSITION 1. *Every operator semi-stable measure on X is infinitely divisible.*

Proof. Let $\mathbf{N}$ be a collection of closed subspaces of X with finite codimension and let $p_N: X \to X/N$, $N \in \mathbf{N}$, be canonical maps.

Let μ be an operator semi-stable measure on X such that

$$[\hat{\mu}(y)]^c = \hat{\mu}(B^* y) e^{i\langle b, y\rangle} \quad \text{for all } y \in X^*,$$

where $B \in B(X)$, $b \in X$ and $c \in (0, 1)$.

Let $\nu = |\mu|^2$ and $N \in \mathbf{N}$. We have

$$\hat{\nu}(y) = \left[\hat{\nu}\left((B^*)^k y\right)\right]^{1/c^k} \quad (y \in X^*)$$

and

$$\hat{\nu}(\pi_N y) = \left[\hat{\nu}\left((B^*)^k \pi_N^* y\right)\right]^{1/c/k} \quad (y \in (X/N)^*)$$

for $k = 1, 2, \ldots$ Let $n_k = [c^{-k}]$ and $y \in (X/N)^*$. If $\hat{\nu}(\pi_N^* y) = 0$, then $\hat{\nu}\left((B^*)^k \pi_N^* y\right) = 0$ for $k = 1, 2, \ldots$

Let $\hat{\nu}(\pi_N^* y) \neq 0$. Since

$$\begin{aligned} \left|\hat{\nu}(\pi_N^* y) - \left[\hat{\nu}\left((B^*)^k \pi_N^* y\right)\right]^{n_k}\right| &\leqslant \left|\left[\hat{\nu}\left((B^*)^k \pi_N^* y\right)\right]^{1/c^k - n_k} - 1\right| \\ &= 1 - \nu(\pi_N^* y)^{c^k(c^{-k} - n_k)} = 1 - \nu(\pi_N^* y)^{1 - c^k[c^{-k}]}, \end{aligned}$$

$\hat{\nu}\left((B^*)^k \pi_N^* y\right)$ converges to $\nu(\pi_N^* y)$. Thus $\pi_N B^k(\nu)^{*n_k} \to \pi_N \nu$. Hence $\pi_N \nu$ is an infinitely divisible measure and $\hat{\mu}(y) \neq 0$ for all $y \in X^*$.

We have

$$\mu(y) = \left[\mu\left((B^*)^k y\right)\right]^{1/c^k} \exp\left(i \frac{1}{c^k} \langle a_k, y\rangle\right) \quad \text{for all } y \in X^*,$$

where

$$a_k = c^k \sum_{j=1}^{k} \frac{1}{c^j} B^{j-1} b \quad (B^0 = I),\ k = 1, 2, \ldots$$

If $N \in \mathbf{N}$, then

$$\hat{\mu}(\pi_N y) = \left[\hat{\mu}\left((B^*)^k \pi_N^* y\right)\right]^{1/c^k} \exp\left(i \frac{1}{c^k} \langle \pi_N a_k, y\rangle\right) \quad \text{for } y \in (X/N)^*.$$

Since

$$\begin{aligned} &\left|\hat{\mu}(\pi_N^* y) - \left[\hat{\mu}\left((B^*)^k \pi_N^* y\right)\right]^{n_k} \exp(i\langle n_k \pi_N a_k, y\rangle)\right| \\ &\qquad \leqslant \left|\left[\hat{\mu}\left((B^*)^k \pi_N y\right) \exp(i\langle \pi_N a_k, y\rangle)\right]^{c^{-k} - n_k} - 1\right| \\ &\qquad = \left|\left[\hat{\mu}(\pi_N^* y)^{c^k}\right]^{1/c^k - n_k} - 1\right| = \left|\hat{\mu}(\pi_N^* y)^{1 - c^k[c^{-k}]} - 1\right|, \end{aligned}$$

$\left\{\left[\hat{\mu}\left((B^*)^k \pi_N^* y\right)\right]^{n_k} \exp(i\langle n_k \pi_N a_k, y\rangle)\right\}$ converges to $\hat{\mu}(\pi_N^* y)$ for every $y \in (X/N)^*$. Thus

$$\pi_N B^k \mu^{*n_k} * \delta_{n_k \pi_N a_k} \to \pi_N \mu$$

and $\pi_N \mu$ is infinitely divisible for all $N \in \mathbf{N}$. By Theorem 1.1.9 of [3], μ is infinitely divisible. This completes the proof of the proposition.

3. Characterization of operator semi-stable measures. The following theorem proves that operator semi-stable probability measures on X are limit laws.

THEOREM 1. *A probability measure μ on X is an operator semi-stable measure if and only if there exist a probability measure ν on X, an operator $B \in B(X)$, sequences $\{a_k\}$ and $\{n_k\}$ of elements of X and of positive integers, respectively, such that, for certain $c \in (0, 1)$,*

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = c \tag{3.1}$$

and

$$B^k \nu^{*n_k} * \delta_{a_k} \to \mu. \tag{3.2}$$

Proof. Necessity. Suppose that μ is an operator semi-stable measure and

$$[\hat{\mu}(y)]^c = \hat{\mu}(B^* y) e^{i\langle b, y\rangle} \quad \text{for all } y \in X^*,$$

where $B \in B(X)$, $c \in (0, 1)$ and $b \in X$.

Let

$$n_k = [c^{-k}] \quad \text{and} \quad a_k = c^k \sum_{j=1}^{k} \frac{1}{c^j} B^{j-1} b \ (B^0 = I).$$

We have

$$\lim_{n\to\infty} \frac{n_k}{n_{k+1}} = c.$$

Since

$$\begin{aligned} \hat{\mu}(y) &= [\hat{\mu}((B^*)^k y)]^{1/c^k} \exp\left(i \frac{1}{c^k} \langle a_k, y\rangle\right) \\ &= [\hat{\mu}((B^*)^k y)]^{n_k} \exp(i\langle n_k a_k, y\rangle) [\hat{\mu}((B^*)^k y) \exp(i\langle a_k, y\rangle)]^{1/c^k - n_k}, \end{aligned} \tag{3.3}$$

the sequence $\{B^k \mu^{*n_k} * \delta_{n_k a_k}\}$ is shift compact (Theorem 3.2.2 of [12]).

By Lemma 1.2.4 of [3] we show that

$$\lim_{k\to\infty} \sup_{y \in U_r^0} |B^k \mu^{*n_k} * \delta_{n_k a_k}(y) - \mu(y)| = 0 \quad \text{for all } r > 0, \tag{3.4}$$

where $U_r = \{x \in X\colon \|x\| \leqslant r\}$ and $U_r^0 = \{y \in X\colon |\langle x, y\rangle| \leqslant 1 \text{ for all } x \in U_r\}$.

We have

$$\begin{aligned} |\hat{\mu}(y) - B^k \mu^{*n_k} * \delta_{n_k a_k}(y)| &\leqslant \left|[\hat{\mu}((B^*)^k y) \exp(i\langle a_k, y\rangle)]^{c^{-k} - [c^{-k}]} - 1\right| \\ &= |[\mu^{c^k}(y)]^{c^{-k} - [c^{-k}]} - 1| = |[\hat{\mu}(y)]^{1 - c^k [c^{-k}]} - 1|. \end{aligned}$$

By $\mu^{1 - c^k [c^{-k}]} \to \delta_0$ (Proposition 1.2 of [8]) and by Lemma 1.2.3 of [3], condition (3.4) holds. Thus the sequence $\{B^k \mu^{*n_k} * \delta_{n_k a_k}\}$ converges to μ.

Sufficiency. Assume that there exist a probability measure ν on X, an operator $B \in B(X)$, sequences $\{a_k\}$ and $\{n_k\}$ of elements of X and of positive integers, respectively, such that (3.1) and (3.2) hold. Further, the sequence

$$\left\{[\hat{\nu}(B^*(B^*)^k y)]^{n_k} \exp(i\langle Ba_k, y\rangle) \exp\left(i\left\langle \frac{n_k}{n_{k+1}} a_{k+1} - Ba_k, y\right\rangle\right)\right\}$$

converges to $[\hat{\mu}(y)]^c$ for all $y \in X^*$. By (3.2) we have

$$B(B^k\nu^{*n_k} * \delta_{a_k}) \to B\mu.$$

Clearly,

$$\hat{\mu}(B^*y)\exp\left(i\left\langle \frac{n_k}{n_{k+1}} a_{k+1} - Ba_k, y\right\rangle\right) \to [\hat{\mu}(y)]^c \quad \text{for all } y \in X^*.$$

By Lemma 1 there exists a $b \in X$ such that $\mu^c = B\mu * \delta_b$, which completes the proof of the theorem.

Given a probability measure μ on X, we denote by $C_p(\mu)$ $(0 < p < \infty)$ the subset of $B(X)$ consisting of all invertible operators A with the property $[\hat{\mu}(y)]^p = A\mu * \delta_a(y)$ for all $y \in X^*$ and certain $a \in X$. Let

$$C(\mu) = \{p \in (0, \infty): C_p(\mu) \neq \emptyset\}.$$

It is clear that if $C(\mu) \neq \{1\}$, then μ is an operator semi-stable measure.

PROPOSITION 2. *Let μ be a probability measure with $C(\mu) \neq \{1\}$. Then either $C(\mu) = \{s^n: n \in Z\}$ for certain $s \in (0, 1)$ or the set $C(\mu)$ is dense in $(0, \infty)$.*

Proof. We assume that $\sup C(\mu) \cap (0, 1) = s < 1$. Suppose that $C_s = \emptyset$. Then there exist $p, q \in C(\mu) \cap (0, 1)$ such that $s^2 < p < q < s$. Further, we get $s < p/q < 1$ and $C_{pq^{-1}} \neq \emptyset$, which contradicts the assumption that s is the supremum of $C(\mu) \cap (0, 1)$. Suppose now that $C(\mu) \neq \{s^n: n \in Z\}$. Then there exists an $r \in C(\mu) \cap (0, s]$ such that $r \neq s^n$ for $n = 1, 2, \ldots$ For some positive integer n_0 we have $s^{n_0+1} < r < s^{n_0}$. Hence $s < r/s^{n_0} < 1$ and $C_{rs^{-n_0}} \neq \emptyset$, which contradicts the assumption that s is the supremum of $C(\mu) \cap (0, 1)$. The proposition is proved.

THEOREM 2. *Let μ be a full probability measure on X. Then there exists an operator $B \in B(X)$ with*

$$\lim_{t \to 0} \exp(B \log t) = 0$$

such that

$$\mu^t = \exp(B \log t)\mu * \delta \qquad \textit{for all } t > 0,$$

where $b_t \in X$, if and only if there exist sequences $\{B_n\}$ and $\{c_n\}$ of operators of the algebra $B(X)$ and of real numbers of $(0, 1)$, respectively, such that

Sem$(\{B_n: n = 1, 2, \ldots\})$ *is compact in the norm topology of* $B(X)$, $c_n \to 1$ *and*

$$\mu^{c_n} = B_n\mu * \delta_{b_n} \quad \textit{for } n = 1, 2, \ldots \textit{ and } b_n \in X.$$

The theorem follows immediately from Theorem 3.1 of [8].

4. Representation of operator semi-stable measures. For the theory of infinitely divisible probability measures on Banach spaces and even on more general algebraic structures we refer to [15] and [3]. In particular, if F is any bounded non-negative Borel measure, then $e(F)$ is defined as

$$e(F) = e^{-F(X)} \sum_{k=0}^{\infty} \frac{1}{k!} F^{*k}, \quad \text{where } F^{*0} = \delta_0.$$

The measure F is called a *Poisson exponent* of $e(F)$.

Let M be a not necessarily bounded Borel measure on X vanishing at 0. If there exists a representation $M = \sup F_n$, where F_n are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of the sequence $\{e(F_n)*\delta_{x_n}\}$ $(x_n \in X)$ is called a *generalized Poisson measure* and is denoted by $\tilde{e}(M)$. Clearly, $\tilde{e}(M)$ is uniquely determined up to translation, i.e. for two cluster points, say μ_1 and μ_2, of $\{e(F_n)*\delta_{x_n}\}$ and $\{e(F_n)*\delta_{y_n}\}$, respectively, we have $\mu_1 = \mu_2 * \delta_x$ for certain $x \in X$ ([15], p. 313). Further, the measure M is called a *generalized Poisson exponent* of $\tilde{e}(M)$. Let $M(X)$ denote the set of all generalized Poisson exponents of X.

By a *Gaussian measure* on X we mean a probability measure ϱ on X such that for every $y \in X^*$ the measure $y\varrho$ induced on the real line is Gaussian. In this paper we consider only symmetric Gaussian measures. For such measures the characteristic functional is of the form

$$\hat{\varrho}(y) = \exp\left(-\frac{1}{2}\langle y, Ry\rangle\right) \quad (y \in X^*),$$

where R is the covariance operator, i.e. a compact operator from X^* into X with the properties $\langle y_1, Ry_2\rangle = \langle y_2, Ry_1\rangle$ for all $y_1, y_2 \in X^*$ (symmetry) and $\langle y, Ry\rangle \geqslant 0$ (non-negativity) (see [17], p. 136, and [2]). By $R(X)$ we denote the set of all covariance operators of Gaussian measures on X.

Tortrat proved in [15] (see also [3]), the following analogue of the Lévy-Khinchine representation of infinitely divisible laws: each infinitely divisible measure μ on X has a unique representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure on X and $M \in M(X)$.

PROPOSITION 3. *Let* $B \in B(X)$. *Then a probability measure* μ *on* X *is operator semi-stable with* $\mu^c = B\mu * \delta_b$ *for some* $c \in (0, 1)$ *and* $b \in X$ *if* *and only if* $\mu = \varrho * \tilde{e}(M)$, *where* ϱ *is a symmetric Gaussian measure with the covariance operator* R *and* $M \in M(X)$ *such that* $cM = BM$ *and* $cR = BRB^*$.

The proof is trivial.

COROLLARY 1. *Let* $B \in B(X)$ *and let* μ *be an operator semi-stable probability measure on* X *with* $\mu^c = B\mu * \delta_b$ *for some* $c \in (0, 1)$ *and* $b \in X$. *If* $\mu = \varrho * \tilde{e}(M)$, *where* ϱ *is a symmetric Gaussian measure and* $M = M(X)$, *then* ϱ *and* M *are concentrated on subspaces* X_1 *and* X_2, *respectively, which are invariant under* B.

Let B be an invertible operator from $B(X)$ with

$$\lim_{n \to +\infty} B^n = 0.$$

Given a subset E of X, we put $\tau(B) = \{B^n x\colon x \in E, n \in Z\}$. It is clear that for any compact set with the property $0 \notin E$ and for any pair r_1, r_2 $(r_1 < r_2)$ of positive numbers the inequality $r_1 \leqslant \|B^{n_k} x_k\| \leqslant r_2$ $(x_k \in X)$ implies the boundedness of the sequence $\{n_k\}$. This simple fact yields the following

LEMMA 2. *Let* E *be a compact subset of* X *and* $0 \notin E$. *Then for every pair* r_1, r_2 $(r_1 \leqslant r_2)$ *of positive numbers the set* $\{x\colon r_1 \leqslant \|x\| \leqslant r_2\} \cap \tau(R)$ *is compact.*

The following lemma reduces our problem of examining a measure $M \in M(X)$ with the property $cM = BM$ for some $c > 0$ to the case of measures concentrated on $\tau(E)$, where E is compact and $0 \notin E$.

LEMMA 3. *Let* $M \in M(X)$ *and* $cM = BM$ *for certain* $c > 0$. *Then there exists a decomposition*

$$M = \sum_{n=1}^{\infty} M_n,$$

where $M_n \in M(X)$, $cM_n = BM_n$, M_n *are concentrated on disjoint sets* $\tau(E_n)$, $0 \notin E_n$ *and* E_n *are compact.*

The lemma follows immediately from Lemma 5.4 of [16].

Now, we are ready to prove the representation of the characteristic functionals of operator semi-stable measures.

THEOREM 3. *Let* B *be an invertible operator from* $B(X)$ *with*

$$\lim_{n \to \infty} B^n = 0.$$

A probability measure μ *on* X *is an operator semi-stable measure and* $\mu^c = B\mu * \delta_b$, *where* $c \in (0, 1)$ *and* $b \in X$, *if and only if there exist an element* $a \in X$, *an operator* $R \in R(X)$ *such that* $cR = BRB^*$ *for certain* $c \in (0, 1)$

and a finite measure λ on $T = \{x \in X: 1 \leqslant \|x\| \leqslant \|B^{-1}\|\}$ such that

$$\hat{\mu}(y) = \exp\Big\{i\langle a, y\rangle - \frac{1}{2}\langle y, Ry\rangle + \tag{4.1}$$

$$+ \sum_{n=-\infty}^{\infty} \frac{1}{c^n} \int_T [\exp(i\langle B^n x, y\rangle) - 1 - i\langle B^n x, y\rangle 1_D(B^n x)]\lambda(dx)\Big\},$$

where 1_D denotes the indicator of the unit ball D in X and $y \in X^*$.

Proof. To prove the necessity let us assume that μ is an operator semi-stable measure, B is an invertible operator from $B(X)$ with

$$\lim_{n\to\infty} B^n = 0$$

and $\mu^c = B\mu * \delta_b$ for certain $c \in (0, 1)$. Further, μ is an infinitely divisible measure and $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure with the covariance operator R and $M \in M(X)$. Moreover, for certain $c \in (0, 1)$ we have

$$BM = cM, \quad cR = BRB^*. \tag{4.2}$$

By Lemma 3 there exists a decomposition

$$M = \sum_{n=1}^{\infty} M_n,$$

where $M_n \in M(X)$, $BM = cM$, M_n are concentrated on disjoint sets $\tau(E_n)$, $0 \notin E_n$ and E_n are compact.

Let $D_n = \tau(E_n) \cap \{x: 1 \leqslant \|x\| \leqslant \|B^{-1}\|\}$. By Lemma 2 the set D_n is compact. We define an equivalence relation in D_n as follows: $x_1 \sim x_2$, $x_1, x_2 \in D_n$, if and only if there exists an integer n such that $x_1 = B^n x_2$. In order to prove the continuity of this relation suppose that $x_n \sim x_n^1$ and that the sequences $\{x_n\}$ and $\{x_n^1\}$ converge to x and x^1, respectively. Then for some integers k_n we have $B^{k_n} x_n = x_n^1$. By the compactness of E_n and the assumption $0 \notin E_n$, the sequence $\{k_n\}$ is bounded. Clearly, for any its cluster point k_0 we have $B^{k_0} x = x^1$, which implies $x \sim x^1$. Thus the relation $\sim$ is continuous. Hence it follows that the quotient space $D_n/\sim$ is compact ([1], p. 97). The coset containing x will be denoted by $[x]$. Further, the mapping $x \to [x]$ from D_n onto $D_n/\sim$ is continuous. A theorem of Kuratowski (Theorem 1.4.2 of [12]) shows that there exists a Borel subset T_n of D_n such that T_n intersects each $[x]$ at exactly one point.

Let f_n be a mapping of $T_n \times Z$ into $\tau(E_n)$ such that $f_n(x, n) = B^n x$. The mapping f_n is continuous and one-one. By a theorem of Kuratowski (Corollary 1.3.2 of [12]) the mapping f_n^{-1} is measurable. Let f be a mapping of $\bigcup_{n=1}^{\infty} T_n \times Z$ into $\bigcup_{n=1}^{\infty} \tau(E_n)$ such that $f(x, m) = f_n(x, m)$ if $x \in T_n$. The mapping f is one-one, and f and f^{-1} are measurable. Hence the σ-field generated by the collection of the sets $B^n(F)$, where n is integer and F stands for Borel subsets of $T_0 = \bigcup_{n=1}^{\infty} T_n$, consists of all Borel subsets of $\bigcup_{n=1}^{\infty} \tau(E_n)$.

Put

$$g(n, F) = M(\{B^n x\colon x \in F\}) \quad (n \in Z). \tag{4.3}$$

Since $BM = cM$, we have

$$g(n, F) = c^{-n} g(0, F) = c^{-n}\lambda_0(F), \tag{4.4}$$

where $\lambda_0(F) = g(0, F)$ for all Borel subsets of T_0. We can extend (4.4) for all Borel subsets of $X \setminus \{0\}$ by the formula

$$M(F) = \sum_{n=-\infty}^{\infty} \frac{1}{c^n} \int_T 1_F(B^n x)\lambda(dx), \tag{4.5}$$

where $\lambda(G) = \lambda_0(G \cap T_0)$ for any Borel subset G of $T = \{x\colon 1 \leqslant \|x\| \leqslant \|B^{-1}\|\}$. Further, from the Dettweiler representation of the characteristic functionals of an infinitely divisible measure on X (Theorem 1.2.5 of [3]) we get the formula

$$\hat{\mu}(y) = \exp\Big\{i\langle a, y\rangle - \frac{1}{2}\langle y, Ry\rangle + \int_X [e^{i\langle x, y\rangle} - 1 - i\langle x, y\rangle 1_D(x)]\, M(dx)\Big\}, \tag{4.6}$$

where $y \in X^*$, $a \in X$, $R \in R(X)$, $M \in M(X)$ and 1_D denotes the indicator of the unit ball D in X. Inserting (4.5) for M into (4.6) we get (4.1).

By a simple calculation we can check that each measure μ with the characteristic functional of form (4.1) fulfils equation (2.1), which completes the proof.

A probability measure μ on X is called *semi-stable* if its characteristic functional satisfies the functional equation

$$[\hat{\mu}(y)]^c = \hat{\mu}(by)e^{i\langle a, y\rangle} \quad \text{for all } y \in X^*, \tag{4.7}$$

where $0 < |b| < 1$, $0 < c < 1$ and $a \in X$.

PROPOSITION 4. *Let μ be a non-degenerate measure on X satisfying* (4.7) *and let p be the unique real solution of the equation $|b|^p = c$. Then*

(a) $0 < p \leqslant 2$;

(b) $p = 2$ *if and only if μ is a Gaussian measure;*

(c) $0 < p < 2$ *if and only if $\mu = \tilde{e}(M)$ for some $M \in M(X)$.*

The proposition is an immediate consequence of the following fact: if μ is a semi-stable measure of X, then $y\mu$ is a semi-stable measure on the real line for all $y \in X^*$.

From now on the unique real solution p of the equation $|b|^p = c$ for a non-degenerate semi-stable probability measure μ on X will be called the *exponent* of μ.

COROLLARY 2. *Let μ be a probability measure on X. Then μ is semi-stable if and only if either μ is Gaussian or there exist constants p $(0 < p < 2)$ and b $(0 < |b| < 1)$, a finite measure λ on $T = \{x\colon 1 \leqslant \|x\| \leqslant 1/|b|\}$ and an element $a \in X$ such that, for every $y \in X^*$,*

$$\hat{\mu}(y) = \exp\Big\{i\langle a, y\rangle + \sum_{n=-\infty}^{\infty} \frac{1}{|b|^{pn}} \int_T [\exp(ib^n\langle x, y\rangle) - 1 - ib^n\langle x, y\rangle 1^D(b^n x)]\lambda(dx)\Big\}, \tag{4.8}$$

where 1_D denotes the indicator of the unit ball D in X.

The measure λ appearing in representation (4.8) will be called the *representing measure* for μ. Let $\Lambda_p(X)$ denote the set of all representing measures corresponding to semi-stable measures on X with the exponent p $(0 < p < 2)$. Clearly, $\lambda \in \Lambda_p(X)$ if and only if the measure M defined by

$$M(F) = \sum_{n=-\infty}^{\infty} \frac{1}{|b|^{pn}} \int_T 1_F(b^n x)\lambda(dx) \tag{4.9}$$

belongs to $M(X)$. The set $M(X)$ has the following property: if N is a non-negative measure on X and $N \leqslant M$, where $M \in M(X)$, then $N \in M(X)$. Hence $\lambda \in \Lambda_p(X)$ if and only if the measure λ_0 defined by $\lambda_0(E) = \lambda(E) + \lambda(-E)$ belongs to $\Lambda_p(X)$. This fact reduces the problem of determining $\Lambda_p(X)$ to examining symmetric measures λ. We say that X is of *type r* $(2 \geqslant r > 0)$ whenever there exists a positive constant c such that for any collection $\xi_1, \xi_2, \ldots, \xi_n$ of independent symmetrically distributed X-valued random variables we have

$$\mathrm{E}\Big\|\sum_{j=1}^{n} \xi_j\Big\|^r \leqslant c \sum_{j=1}^{n} \mathrm{E}\|\xi_j\|^r.$$

THEOREM 4. *If X is of type r and $r > p$, then $\Lambda_p(X)$ consists of all finite Borel measures on T.*

Proof. We use arguments similar to those given by Jurek and Urbanik in [7]. To prove the theorem it suffices to show that for each symmetric finite measure λ on X the measure M defined by (4.9) belongs to $M(X)$. Let

$$M_0(F) = \sum_{n=-\infty}^{0} \frac{1}{|b|^{pn}} \int_T 1_F(b^n x)\lambda(dx)$$

and

$$M_k(F) = \frac{1}{|b|^{pk}} \int_T 1_F(b^k x)\lambda(dx) \qquad (k = 1, 2, \ldots);$$

then the measures M_n $(n = 0, 1, 2, \ldots)$ are finite on X and vanish at 0. Put, for simplicity, $\mu_k = e(M_k)$ $(k = 0, 1, \ldots)$. Since

$$M = \sum_{k=0}^{\infty} M_k,$$

we conclude that $M \in M(X)$ if and only if the sequence $\{\mu_0 * \mu_1 * \ldots * \mu_n\}$ converges to a probability measure on X or, equivalently, the series $\sum_{k=0}^{\infty} \eta_k$ of independent X-valued random variables $\eta_0, \eta_1, \ldots$ with probability distributions $\mu_0, \mu_1, \ldots$, respectively, converges almost surely (Theorem 3.1 of [4]). To prove that $\sum_{k=0}^{\infty} \eta_k$ converges almost surely, it suffices, by the Borel-Cantelli lemma, to show the convergence of the series

$$\sum_{k=0}^{\infty} \mu_k(\{x\colon \|x\| > a^k\}), \tag{4.10}$$

where $a = |b|^{(r+1)^{-1}(r-p)} < 1$. Setting $a_k = M_k(X)$ and $\nu_k = a_k^{-1} M$ for $k = 1, 2, \ldots$, we get

$$\mu_k = \exp(-a_k) \sum_{n=0}^{\infty} \frac{a_k^n}{n!} \nu_k^{*n} \tag{4.11}$$

and

$$a_k = |b|^{-pk}\lambda(T).$$

Further, for a positive constant c we obtain

$$\int_X \|x\|^r \nu_k^{*n}(dx) \leqslant c_1 |b|^{kr} n.$$

Consequently, by (4.11) we have

$$\int_X \|x\|^r \mu_k(dx) \leqslant c_1 |b|^{kr} \exp(-a_k) \sum_{n=0}^{\infty} \frac{a_k^n}{(n-1)!}.$$

Since

$$\exp(-a_k)\sum_{n=0}^{\infty}\frac{a_k^n}{(n-1)!}\leqslant c_2 a_k \quad \text{for certain } c_2>0,$$

we get the inequality

$$\int_X \|x\|^r \mu_k(dx)\leqslant c_3 a^{k(r+1)} \quad (k=1,2,\ldots)$$

with a constant c_3. Consequently,

$$\mu_k(\{x\colon \|x\|>a^k\})\leqslant a^{-kr}\int_X \|x\|^r\mu_k(dx)\leqslant c_3 a^k \quad (k=1,2,\ldots),$$

which proves the convergence of series (4.10). This completes the proof of the theorem.

In particular, from Theorem 4 for $p<1$ and every Banach space X as well as for $1\leqslant p<r$ and Banach spaces X of type r we get the description of $\Lambda_p(X)$.

REFERENCES

[1] N. Bourbaki, *Eléments de mathématique, I. Les structures fondamentales de l'analyse*, Livre III, *Topologie générale*, Paris 1951.

[2] В. И. Тариеладзе и С. А. Чобанян, *О полной непрерывности ковариационного оператора*, Сообщения Академии наук Грузинской ССР 70 (1973), p. 273-276.

[3] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmaße auf Badrikianschen Räumen*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 34 (1976), p. 285-311.

[4] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka Journal of Mathematics 5 (1968), p. 35-48.

[5] R. Jajte, *On stable distribution in Hilbert space*, Studia Mathematica 30 (1968), p. 63-71.

[6] — *Semi-stable probability measures on R^N*, ibidem 61 (1977), p. 29-39.

[7] Z. Jurek and K. Urbanik, *Remarks on stable measures on Banach spaces*, Colloquium Mathematicum 38 (1978), p. 269-276.

[8] W. Krakowiak, *Operator-stable probability measures on Banach spaces*, ibidem 41 (1979), p. 313-326.

[9] В. М. Круглов, *Об одном расширении класса устойчивых распределений*, Теория вероятностей и ее применения 17 (1972), p. 723-732.

[10] A. Kumar, *Semi-stable probability measures on Hilbert spaces*, Journal of Multivariate Analysis 6 (1976), p. 309-318.

[11] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1937.

[12] K. R. Parthasarathy, *Probability measures on metric spaces*, New York 1967.

[13] R. N. Pillai, *Semi-stable laws as limit distributions*, The Annals of Mathematical Statistics 42 (1971), p. 780-783.

[14] C. R. Rao and B. Ramachandran, *Some results on characteristic functions and characterization of the normal and generalized stable laws*, Sānkhya, Series A, 30 (1968), p. 125-140.

[15] A. Tortrat, *Structure des lois indéfiniment divisibles dans un espace vectoriel topologique (séparé) X*, Symposium on Probability Methods in Analysis, Lecture Notes in Mathematics 31, Berlin 1967, p. 299-328.

[16] K. Urbanik, *Lévy's probability measures on Banach spaces*, Studia Mathematica (to appear).

[17] Н. Н. Вахания, *Вероятностные распределения в линейных пространствах*, Тбилиси 1971.

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