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ON A CLASS OF MIXED BOUNDARY VALUE PROBLEMS FOR LINEAR HYPERBOLIC EQUATIONS

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In this paper we are concerned with the initial boundary value problem for linear hyperbolic equation of order m defined in the Euclidean space R^{n+1}. The boundary conditions are given on a hyperplane of co-dimension $\nu \geqslant 1$. The problems of this sort were studied first by Sobolev [3] for polyharmonic equations and investigated later by Sternin [4] for elliptic equations of arbitrary order. Sternin introduced an analogue of the Lopatinskiĭ matrix and proved that the problem has finite index if this matrix is non-singular.

In this paper we introduce the matrix $\hat{D}(\zeta)$ (Section 7) similar to that used by Sternin, but containing the symbol of the operator P deviated into complex domain. Assuming that it may be estimated from below by some power of $\operatorname{Im}\zeta$ (see (7.2)) we prove that our problem is properly posed in terms of suitably defined generalized Sobolev spaces (Theorem 4).

The method of solving the problem was stimulated by the paper of Chazarain and Piriou [1]. In particular, their lemma (quoted as Theorem I in this paper) is the essential tool for constructing the linear convolution system (Λ) (Section 6), from which the unknown densities c_a of the potentials u_a may be computed.

In this paper we restrict ourselves to the simplest case where the differential operators under consideration have constant coefficients and consist only of terms of higher order. In the last section we give a simple example of a boundary value problem satisfying our assumptions.

1. Basic definitions and notation. In this section we gather the notation which will be used in the sequel:

$x, \xi \in R^{\nu}$, $y, \eta \in R^{n-\nu}$, $t, \tau \in R$;

$(y, t) = s$, $(\eta, \tau) = \sigma \in R^{n+1-\nu}$;

$(x, y, t) = (x, s)$, $(\xi, \eta, \tau) = (\xi, \sigma) \in R^{n+1}$;

$n_0 = (0, 1) \in R^{n+1-\nu}$ is the versor of t-axis;

$\zeta = \sigma + in \in C^{n+1-\nu}$, $\zeta_0 = \sigma - in_0$;

$\partial_{x_j} = \partial/\partial_{x_j}$, $D_{x_j} = i^{-1}\partial_{x_j}$, $D_x = (D_{x_1}, \ldots, D_{x_\nu})$;
$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_\nu}^{\alpha_\nu}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_\nu}^{\alpha_\nu}$ $(\alpha = (\alpha_1, \ldots, \alpha_\nu)$, $|\alpha| = \alpha_1 + \ldots + \alpha_\nu)$;
similarly ∂_s^β, D_s^β, D_s, etc.

The Fourier transform of a tempered distribution g will be denoted by $\hat{g}$ or Fg; $F^{-1}g$ stands for the inverse Fourier transform.

For an arbitrary open cone $C \subset R^p$ the dual cone C^* is defined as

$$C^* = \{z \in R^p : (z, z') \geqslant 0 \text{ for } z' \in C\}.$$

The space of tempered distributions will be denoted, as usual, by $S'(R^p)$. By $O_M(R^p)$ we mean the set of all functions infinitely differentiable in R^p with all the derivatives of polynomial growth at infinity.

We also put

$$H_r = \Big\{\varphi \in S'(R^{n+1-\nu}) \colon \|\varphi\|_r^2 \overset{\text{df}}{=} \int (1+|\sigma|^2)^r |\hat{\varphi}(\sigma)|^2 d\sigma < \infty\Big\},$$

$$H_{k,r} = \Big\{\varphi \in S'(R^{n+1}) \colon$$
$$\|\varphi\|_{k,r}^2 \overset{\text{df}}{=} \iint (1+|\xi|^2+|\sigma|^2)^k (1+|\sigma|^2)^r |\hat{\varphi}(\xi, \sigma)|^2 d\xi d\sigma < \infty\Big\},$$

$$S'_+ = \big\{\varphi \in D'(R^p) \colon \operatorname{supp}\varphi \subset \{t \geqslant 0\},\ e^{-t}\varphi \in S'(R^p)\big\}$$

for $p = n+1$ or $p = n+1-\nu$.

The properties of the spaces H_r and $H_{k,r}$ with $\nu = 1$ are given in detail in [2]. Throughout this paper we shall use their slight modifications

$$H_r^+ = \{\varphi \in S'_+ : e^{-t}\varphi \in H_r\} \quad \text{and} \quad H_{k,r}^+ = \{\varphi \in S'_+ : e^{-t}\varphi \in H_{k,r}\}$$

with the norms

$$|\varphi|_r \overset{\text{df}}{=} \|e^{-t}\varphi\|_r \quad \text{and} \quad |\varphi|_{k,r} \overset{\text{df}}{=} \|e^{-t}\varphi\|_{k,r},$$

respectively.

We shall deal with differential operators of the form $P(D_x, D_y, D_t)$, where $P(\xi, \eta, \tau)$ is a homogeneous polynomial with real coefficients. We say that the operator P (or the polynomial P) is ***hyperbolic*** with respect to t if

(a) $P(0, n_0) \neq 0$,

(b) for arbitrary fixed ξ and η the equation $P(\xi, \eta, \tau) = 0$ has only real (not necessarily distinct) τ-roots.

The properties of hyperbolic operators used in the sequel can be found in [2]. We shall deal particularly with the fundamental solution E

of the operator P defined by

$$\langle E, \check{\varphi}\rangle = (2\pi)^{-n-1} \int\int \frac{\hat{\varphi}(\xi, \zeta_0)}{P(\xi, \zeta_0)} d\xi d\sigma,$$

where $\varphi \in C_0^\infty(R^{n+1})$ and $\check{\varphi}(x, s) \overset{\mathrm{df}}{=} \varphi(-x, -s)$.

It is known that $E \in S'_+(R^{n+1})$ and

$$F(e^{-t}E)(\xi, \sigma) = \frac{1}{P(\xi, \zeta_0)}. \tag{1.1}$$

Moreover, the support of E is contained in the dual cone Γ^*, where Γ is the cone of hyperbolicity of the operator P. Put $\Gamma_0 = \Gamma \cap \{x = 0\}$. We shall use the estimate

$$|P(\xi, \zeta)| \geqslant |P(0, n)| \tag{1.2}$$

with $n \in -\Gamma_0$ (see [1] and [2]).

2. Preliminary results. The main tool in our investigations is the following

THEOREM I (Chazarain, Piriou [1]). *Let Γ_1 be an open convex cone in R^p. Then the following statements are equivalent:*

(A) *F is a holomorphic function in $R^p - i\Gamma_1$, homogeneous of degree $-p-d$, with the following property: for every closed convex cone $L \subset \Gamma_1$ there are a real number θ and a positive number c such that*

$$|F(\zeta)| \leqslant c|\operatorname{Im}\zeta|^\theta \tag{2.1}$$

for $\zeta \in R^p - iL$, $|\zeta| = 1$, $\operatorname{Im}\zeta \neq 0$.

(B) *There is a tempered distribution f, homogeneous of degree d, with* $\operatorname{supp} f \subset \Gamma_1^*$ *and such that*

$$F(\zeta) = [e^{-(z, \operatorname{Im}\zeta)} f(z)]^\wedge \quad (\zeta \in R^p - i\Gamma_1).$$

We shall use mainly the fact that (A) implies (B).

We prove now

PROPOSITION 1. *Let $C \subset R^p$ be an open convex cone containing the positive half of z_p-axis and let $f, g \in D'(R^p)$ satisfy the conditions*

$$\operatorname{supp} f \subset C^* \quad \textit{and} \quad \operatorname{supp} g \subset \{z_p \geqslant 0\}.$$

Then

$$e^{-z_p}(f * g) = e^{-z_p} f * e^{-z_p} g. \tag{2.2}$$

Proof. It is sufficient to show that the convolutions on both sides of (2.2) are well defined. Obviously, for arbitrary $j = 1, \ldots, p-1$ and sufficiently small $\varepsilon > 0$ the points $(0, \ldots, 0, \pm\varepsilon, 0, \ldots, 0, 1)$ belong to C (here the j-th coordinate is $\pm\varepsilon$). Therefore, for every $z \in C^*$ we have

$$(2.3) \qquad z_p \geqslant \varepsilon |z_j| \quad (j = 1, \ldots, p-1).$$

It follows from (2.3) that each $z \in C^*$ may be written in the form $z = |z|\mathring{z}$, where $|\mathring{z}| = 1$ and $\mathring{z}_n \geqslant c > 0$ (with a constant c independent of z). From this statement it may be easily deduced that the sets $\operatorname{supp}\varphi(z+z')$ and $\operatorname{supp} f \times \operatorname{supp} g$ (considered as the subsets of R^{2p}) have a compact intersection for an arbitrary fixed $\varphi \in D(R^p)$. Thus the convolution on the left-hand side of (2.2) is well defined. The right-hand side can be treated similarly.

PROPOSITION 2. *Let $f, g \in S'(R^p)$ and $f \in O_M(R^p)$. Then $f*g \in S'(R^p)$ and $(f*g)^{\wedge} = \hat{f}\hat{g}$.*

This follows immediately from Theorems 30.3 and 30.4 in [5].

PROPOSITION 3. *Let $P(\xi, \sigma)$ be a homogeneous polynomial of degree m with real coefficients such that $P(\xi, 0) > 0$ for $\xi \neq 0$. Then for an arbitrary compact $K \subset C^{n+1-\nu}$ there exist positive constants B and c such that*

$$(2.4) \qquad |P(\xi, \zeta)| \geqslant c|\xi|^m$$

for $\zeta \in K$ and $|\xi| \geqslant B$.

Proof. We have

$$P(\xi, \zeta) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha + \sum_{|\alpha|<m} b_\alpha(\zeta)\xi^\alpha,$$

where b_α are polynomials of degree $m-|\alpha|$. Therefore

$$(2.5) \qquad |P(\xi, \zeta)| \geqslant d|\xi|^m - M|\xi|^{m-1},$$

where

$$d = \inf_{|\xi|=1} \sum_{|\alpha|=m} a_\alpha \xi^\alpha \quad \text{and} \quad M = \sup_{\zeta\in K} \sum_{|\alpha|<m} |b_\alpha(\zeta)|.$$

Now (2.5) yields (2.4) with $c = d/2$ and $B = 2M/d$.

PROPOSITION 4. *Let P be a hyperbolic polynomial satisfying assumptions of Proposition 3 and let*

$$F_{\beta,n}(\sigma) = \int_{R^\nu} \frac{\xi^\beta}{P(\xi, \sigma + in)} d\xi$$

with arbitrary fixed $n \in -\Gamma_0$ and multi-index $\beta \in R^\nu$. Suppose that $m > \nu$ and $|\beta| < m-\nu$. Then $F_\beta \in O_M(R^{n+1-\nu})$.

Proof. It is easy to show by induction that for an arbitrary multi-index $\varrho \in R^{n+1-\nu}$ ($|\varrho| > 0$) we have

$$\partial_\sigma^\varrho \frac{1}{P(\xi, \sigma+in)} = \frac{w(\xi, \sigma, n)}{[P(\xi, \sigma+in)]^{|\varrho|+1}},$$

where w is a polynomial of degree at most $|\varrho|m-1$ with respect to ξ. Therefore, it follows from Proposition 3 that all the integrals of the form

$$\int_{R^\nu} \partial_\sigma^\varrho \left(\frac{\xi\beta}{P(\xi, \sigma+in)}\right) d\xi$$

converge uniformly with respect to σ lying in an arbitrarily fixed compact in $R^{n+1-\nu}$. Thus $F_{\beta,n}$ is in $C^\infty(R^{n+1-\nu})$ and may be differentiated under the sign of the integral. For $\zeta = \sigma + in$ we put $F_\beta(\zeta) = F_{\beta,n}(\sigma)$. Then F_β and all the derivatives $\partial_\sigma^\varrho F_\beta$ are homogeneous functions. To prove our statement it is therefore sufficient to estimate the function

$$(\partial_\sigma^\varrho F_\beta)\left(\frac{\zeta}{|\zeta|}\right) = \int_{R^\nu} \frac{w(\xi, \sigma/|\zeta|, n/|\zeta|)}{[P(\xi, \zeta/|\zeta|)]^{|\varrho|+1}}\, d\xi. \tag{2.6}$$

According to Proposition 3 the integral on the right-hand side may be written as the sum of two integrals $\int_{|\xi|\leqslant B}$ and $\int_{|\xi|\geqslant B}$, which we denote by K_B and L_B, respectively. Then using (1.2) we get

$$|K_\beta(\zeta)| \leqslant c|\zeta|^{m(|\varrho|+1)} \int_{|\xi|\leqslant B} w\left(\xi, \frac{\sigma}{|\zeta|}, \frac{n}{|\zeta|}\right) d\xi, \tag{2.7}$$

where $c = [P(0, n)]^{-|\varrho|-1}$. The integral on the right-hand side of (2.7) is obviously a bounded function of $\sigma \in R^{n+1-\nu}$; using once more Proposition 3 we see that the integral L_B has the same property.

3. Some properties of the distributions supported by the hyperplane $x = 0$.

THEOREM 1. *The following statements are equivalent:*

(a) $g \in H_{k,r}$ *and* supp g *is contained in the hyperplane* $x = 0$.

(b) *If* $k < -\nu/2$, *then*

$$g = \sum_{|\alpha|<-k-\nu/2} d_\alpha(s) D^\alpha \delta(x), \tag{3.1}$$

where

$$d_\alpha \in H_{k+r+\nu/2+|\alpha|}; \tag{3.2}$$

if $k \geqslant -\nu/2$, *then* g *vanishes identically*.

Proof. Suppose that (b) holds. If $g = 0$, then (a) is obvious, and so it is sufficient to consider the case $k < -\nu/2$. Putting

$$(3.3)\qquad d_\alpha(s)D^\alpha\delta(x) = g_\alpha(x, s),$$

we have $\hat{g}_\alpha(\xi, \sigma) = \hat{d}_\alpha(\sigma)\xi^\alpha$ and, therefore,

$$\|g_\alpha\|_{k,r}^2 = \int\limits_{R^{n+1-\nu}}\int\limits_{R^\nu}(1+|\xi|^2+|\sigma|^2)^k(1+|\sigma|^2)^r\xi^{2\alpha}|\hat{d}_\alpha(\sigma)|^2d\xi d\sigma.$$

Setting $\xi = (1+|\sigma|^2)^{1/2}\eta$, we obtain

$$(3.4)\qquad \|g_\alpha\|_{k,r}^2 = \|d_\alpha\|_{k+r+\nu/2+|\alpha|}^2\int\limits_{R^\nu}(1+|\eta|^2)^k\eta^{2\alpha}d\eta.$$

As the integral on the right-hand side of (3.4) converges, $g_\alpha \in H_{k,r}$, and so $g \in H_{k,r}$ according to (3.1) and (3.3).

To prove the converse implication we first show the following:

(I) *Let p be a fixed non-negative integer and r an arbitrary real number. Then the functional*

$$(3.5)\qquad H_{p,r}\ni f \to \Big[\sum_{0\leqslant|\alpha|\leqslant p}\iint(1+|\sigma|^2)^{p+r-|\alpha|}|D_x^\alpha\hat{f}(x, \sigma)|^2dxd\sigma\Big]^{1/2}$$

defines a norm which is equivalent to the usual norm $\|\cdot\|_{p,r}$.

Indeed, it suffices to estimate the functional (3.5) for $f \in S$. It is easy to verify that

$$(3.6)\qquad \|f\|_{p,r}^2 = \|f\|_{p-1,r+1}^2+\sum_{j=1}^{\nu}\|D_{x_j}f\|_{s-1,r}^2.$$

Using (3.6) p times we obtain

$$(3.7)\qquad \|f\|_{p,r}^2 = \sum_{0\leqslant|\alpha|\leqslant p} n_\alpha\|D_x^\alpha f\|_{0,p+r-|\alpha|}^2,$$

where n_α are some positive integers. Using Parseval's equality in $L^2(R_\nu)$ we get (for arbitrary $g \in S$ and real r)

$$\|g\|_{0,r}^2 = \iint(1+|\sigma|^2)^r|\hat{g}(x, \sigma)|^2dxd\sigma.$$

Replacing g by $D_x^\alpha f$ and r by $p+r-|\alpha|$ and using (3.7) we obtain (I). Suppose now that (a) holds. Then g is of the form

$$(3.8)\qquad g = \sum_{|\alpha|<a} d_\alpha(s)D^\alpha\delta(x)$$

with some $d_a \in D'(R^{n+1-\nu})$ (the sum is finite because g is a tempered distribution, hence of finite order).

It remains to show that (3.2) holds. Let us choose a function $\psi \in C^\infty(R^\nu)$ vanishing outside the unit sphere and equal identically to one in a neighbourhood of the origin. For fixed β $(0 \leqslant |\beta| \leqslant q)$ and for any $\varphi_\beta \in C_0^\infty(R^{n+1-\nu})$ we put

$$\hat{\varphi}(x, \sigma) = \psi(\lambda x)\hat{\varphi}_\beta(\sigma)\frac{x^\beta}{\beta!} \tag{3.9}$$

with $\lambda = (1+|\sigma|^2)^{1/2}$. It is easy to verify that the right-hand side of (3.9) belongs to $S_{x,\sigma}$. Therefore, φ is well defined and

$$D_x^a \varphi(0, s) = \begin{cases} 0 & \text{for } a \neq \beta, \\ \varphi_\beta(\sigma) & \text{for } a = \beta. \end{cases} \tag{3.10}$$

Let now $\delta \geqslant 0$ be chosen so that $-k+\delta$ is a non-negative integer. Using (I) with $p = -k+\delta$ we have

$$\|\varphi\|^2_{-k+\delta,-r-\delta} \leqslant c \sum_{0 \leqslant |\gamma| \leqslant -s+\delta} \int_{R^{n+1-\nu}} \lambda^{-2(k+r+|\gamma|)} \int_{R^\nu} |D_x^\gamma \hat{\varphi}(x, \sigma)|^2 dx\, d\sigma. \tag{3.11}$$

But

$$D_x^\gamma \hat{\varphi}(x, \sigma) = \frac{\lambda^{|\gamma|-|\beta|}}{\beta!} \hat{\varphi}_\beta(\sigma)\, \chi_{\gamma,\beta}(\lambda x), \tag{3.12}$$

where $\chi_{\gamma,\beta}(y) = D^\gamma[y^\beta \psi(y)]$ $(y \in R^\nu)$. Substituting $y = \lambda x$, by (3.12) we obtain

$$\int_{R^\nu} |D_x^\gamma \hat{\varphi}(x, \sigma)|^2 dx = c_{\gamma,\beta} \lambda^{2|\gamma|-2|\beta|-\nu} |\hat{\varphi}_\beta(\sigma)|^2 \tag{3.13}$$

with

$$c_{\gamma,\beta} = (\beta!)^{-2} \int_{R^\nu} |\chi_{a,\beta}(y)|^2 dy.$$

Now (3.11) and (3.13) yield

$$\|\varphi\|^2_{-k+\delta,-r-\delta} \leqslant c \|\varphi_\beta\|^2_{-k-r-|\beta|-\nu/2}. \tag{3.14}$$

It follows from (3.8) and (3.10) that

$$\langle g, \varphi \rangle = (-1)^{|\beta|} \langle d_\beta, \varphi_\beta \rangle. \tag{3.15}$$

As $g \in H_{k-\delta,r+\delta}$, we can write

$$|\langle g, \varphi \rangle| \leqslant \|g\|_{k-\delta,r+\delta} \|\varphi\|_{-k+\delta,-r-\delta},$$

which, together with (3.14) and (3.15), yields

$$|\langle d_\beta, \varphi_\beta\rangle| \leqslant c\,\|\varphi_\beta\|_{-k-r-|\beta|-\nu/2}. \tag{3.16}$$

Since φ_β was arbitrarily chosen, (3.2) follows from (3.16).

To complete the proof it remains to estimate the number q occurring in (3.8). Without loss of generality we may assume that d_a does not vanish for some a with $|a| = q$. Now we have

$$|\hat{g}(\xi, \sigma)|^2 = Q(\sigma; \xi) + \sum_{|a|+|\beta|<2q} \hat{d}_a(\sigma)\overline{\hat{d}_\beta}(\sigma)\,\xi^{a+\beta}, \tag{3.17}$$

where

$$Q(\sigma; \xi) = \sum_{|a|=|\beta|=q} \hat{d}_a(\sigma)\overline{\hat{d}_\beta}(\sigma)\,\xi^{a+\beta}.$$

In the sequel of the proof we need the following:

(II) *Let*

$$A = \{\sigma\colon\ Q(\sigma; \cdot) \textit{ does not vanish identically in } \xi\}.$$

Then $\operatorname{mes} A > 0$.

(III) *For an arbitrary fixed $\sigma \in A$ there exists a cone $\Lambda_\sigma \subset R^\nu$ and a positive number a_σ such that*

$$|\hat{g}(\xi, \sigma)|^2 \geqslant a_\sigma |\xi|^{2q} \qquad (\xi \in \Lambda_\sigma). \tag{3.18}$$

To prove (II), suppose that $\operatorname{mes} A = 0$. Then $Q(\sigma; \cdot)$ vanishes identically in ξ for almost all σ. But

$$Q(\sigma; \xi) = \Big|\sum_{|a|=q} \hat{d}_a(\sigma)\,\xi^a\Big|^2;$$

therefore, for all $|a| = q$ the Fourier transform $\hat{d}_a$ vanishes almost everywhere. Thus all the distributions d_a ($|a| = q$) vanish in the sense of S', which yields a contradiction and completes the proof of (II).

To prove (III) let us put $\xi = t\eta$ with $t = |\xi|$ and $|\eta| = 1$. Then

$$|\hat{g}(\xi, \sigma)|^2 = t^{2q} Q(\sigma; \eta) + \sum_{|a|+|\beta|<2q} \hat{d}_a(\sigma)\overline{\hat{d}_\beta(\sigma)}\,t^{|a|+|\beta|}\,\eta^{a+\beta}. \tag{3.19}$$

As $Q(\sigma; \cdot)$ is a continuous non-negative function which does not vanish identically, there is a domain Ξ_σ of the unit sphere such that

$$Q(\sigma; \eta) \geqslant c_\sigma \qquad (\eta \in \Xi_\sigma) \tag{3.20}$$

with some positive constant c_σ. According to (3.19) and (3.20) we have

$$|\hat{g}(\xi, \sigma)|^2 \geqslant c_\sigma t^{2q} - M_\sigma t^{2k+1}, \tag{3.21}$$

where M_σ depends on $\max\limits_{0\leqslant|\alpha|\leqslant q} |\hat{d}_\alpha(\sigma)|$. Putting

$$A_\sigma = \{\xi \in R^\nu : t \geqslant 2M_\sigma,\ \eta \in \Xi_\sigma\},$$

we get from (3.21) the inequality (3.18) with $a_\sigma = c_\sigma/2$.

Now we can complete the proof of the theorem. It follows from (a) that the integral

$$\int\limits_{R^\nu} (1+|\xi|^2+|\sigma|^2)^k (1+|\sigma|^2)^r |\hat{g}(\xi,\sigma)|^2 d\xi$$

converges for almost all σ, particularly, by (II), for some $\sigma = \sigma_0 \in A$. Then in virtue of (III) the integral

$$J_0 = \int\limits_{A_{\sigma_0}} (1+|\xi|^2+|\sigma_0|^2)^k |\xi|^{2q} d\xi$$

is also convergent. Introducing the spherical coordinates in R^ν we see that the convergence of J_0 is equivalent to the convergence of the integral

$$\int\limits_1^\infty t^{2k+2q+\nu-1} dt;$$

therefore we have $0 \leqslant q < -k-\nu/2$, which yields (b). The proof is complete.

THEOREM 2. *Suppose that $g \in H_{k,r}$ with $k < -\nu/2$ and* supp g *is contained in the hyperplane $x = 0$. Then the following statements are equivalent:*

(a) *g vanishes in the half-space $t < 0$.*

(b) *All d_α occurring in* (3.1) *vanish for $t < 0$.*

Proof. It is evident that (b) implies (a). To prove the converse let us put, for fixed β $(0 \leqslant |\beta| < -k-\nu/2)$,

$$\psi(x, s) = \frac{x^\beta}{\beta!} \varphi_\beta(s)$$

with arbitrarily chosen $\varphi_\beta \in C_0^\infty(R^{n+1-\nu})$. Obviously,

$$D_x^\alpha \psi(0, s) = \begin{cases} 0 & \text{for } \alpha \neq \beta, \\ \varphi_\beta(s) & \text{for } \alpha = \beta, \end{cases}$$

and therefore, according to (3.1),

$$\langle g, \varphi\rangle = (-1)^{|\beta|} \langle d_\beta, \varphi_\beta\rangle.$$

It is evident that φ_β vanishes for $t \geqslant -\delta$ (with some $\delta > 0$) if and only if so does ψ. Therefore (a) implies (b).

From Theorems 1 and 2 we have immediately

COROLLARY 1. *The following statements are equivalent:*

(a) $q \in H^+_{k,r}$ *and* supp g *is contained in the hyperplane* $x = 0$.

(b) *If* $k < -\nu/2$, *then* g *is of the form*

$$g = \sum_{|\alpha| < -k-\nu/2} c_\alpha(s) D^\alpha \delta(x), \qquad \textit{where } c_\alpha \in H^+_{k+r+\nu/2+|\alpha|};$$

if $k \geqslant -\nu/2$, *then* g *vanishes.*

Remark. In the special case where $\nu = n+1$ (so the distributions under considerations do not depend on s) Theorem 1 (without proof) can be found in [4].

4. Formulation of the boundary value problem. Let us now consider the following differential operators with constant coefficients: $P(D_x, D_y, D_t)$ of order m and $B_\beta(D_x, D_y, D_t)$ of order m_β, where β is a multi-index in R^ν. We make the following assumptions:

(A_1) P is hyperbolic with respect to t.

(A_2) $P(\xi, 0, 0) \neq 0$ for $\xi \neq 0$ (this means that the hyperplane $x = 0$ is not characteristic with respect to P).

(A_3) The operators P and B_β consist only of terms of the highest order (so the polynomials P and B_β are homogeneous).

We shall deal with the following boundary value problem $(\pi_{k,r})$: Find a distribution u satisfying the conditions

$$u \in H^+_{k,r}, \tag{4.1}$$

$$P(D_x, D_y, D_t)u = 0 \quad \text{for } x \neq 0, \tag{4.2}$$

$$B_\beta(D_x, D_y, D_t)u|_{x=0} = g_\beta \quad (|\beta| < l) \tag{4.3}$$

with given g_β. The number l of boundary conditions will be defined in the sequel. The boundary value in (4.3) is to be understood in the sense of trace, therefore we have to assume

$$k > m_\beta + \nu/2. \tag{4.4}$$

Equation (4.2) can be written in another form, namely

$$P(D_x, D_y, D_t u) = f, \tag{4.5}$$

where f is in $H^+_{k-m,r}$ and vanishes in the complement of the hyperplane $x = 0$.

PROPOSITION 5. *Every solution of* $(\pi_{k,r})$, *if exists, is of the form*

$$u = f * E. \tag{4.6}$$

Proof. It follows from Proposition 1 that the convolution $u*E$ is well defined. According to the well-known properties of the convolution product we have $P(u*E) = u*PE$ and, otherwise, $P(u*E) = Pu*E$. Since $PE = \delta$, we obtain (4.6). The proof is complete.

In virtue of Corollary 1 we have now two possibilities. If $k \geqslant m-\nu/2$, then f vanishes identically, and so does u according to (4.6). In this case $(\pi_{k,r})$ has no solution if the boundary data g_β do not vanish. In the other case, where

$$k < m-\nu/2, \tag{4.7}$$

we have

$$f(x, s) = \sum_{|\alpha|<l_k} c_\alpha(s) D^\alpha \delta(x),$$

where $c_\alpha \in H^+_{k+r-m+\nu/2+|\alpha|}$ and $l_k = m-k-\nu/2$.

According to Proposition 5 we shall seek the solution in the form

$$u(x, s) = \sum_{|\alpha|<l_k} c_\alpha(s) D^\alpha \delta(x) * E(x, s), \tag{4.8}$$

where the unknown distributions c_α are to be defined from the boundary conditions (4.3). Therefore, it is natural to put $l = l_k$.

5. Certain integral formulas. In this section we obtain integral formulas for the convolutions occurring on the right-hand side of (4.8) and for their derivatives. Introducing the notation

$$u_\alpha(x, s) = c_\alpha(s) D^\alpha \delta(x) * E(x, s),$$

we have

PROPOSITION 6. *Suppose* $c_\alpha \in S'_+(R^{n+1-\nu})$. *Then*

$$e^{-t} D^\gamma_x u_\alpha(x, s) = F^{-1}\left[\frac{\xi^{\alpha+\gamma}\hat{c}_\alpha(\zeta_0)}{P(\xi, \zeta_0)}\right]. \tag{5.1}$$

This follows easily from Propositions 1 and 2 in virtue of (1.1) and (1.2).

PROPOSITION 7. *Let*

$$G_\beta(\zeta) = \int_{R^\nu} \frac{|\xi^\beta|}{|P(\xi, \zeta)|}\, d\xi \quad (\zeta \in R^{n+1-\nu} - i\Gamma_0)$$

and

$$d_\beta = |\beta| - m + \nu.$$

If $d_\beta < 0$, *then*

(a) G_β *is a continuous function, homogeneous of degree* d_β;

(b) *for* $\zeta = \zeta_0$ *the estimate*

$$G_\beta(\zeta_0) \leqslant c(1+|\sigma|^2)^{(|\beta|+\nu)/2} \tag{5.2}$$

holds.

Proof. Statement (a) follows immediately from Proposition 3.

To prove (b) it is sufficient to estimate G_β on the unit sphere in $C^{n+1-\nu}$. According to Proposition 3 we have

$$\int\limits_{|\xi|\geqslant B} \frac{|\xi^\beta|}{P(\xi, \zeta_0/|\zeta_0|)}\, d\xi \leqslant c_1,$$

and (1.2) yields

$$\int\limits_{|\xi|\leqslant B} \frac{|\xi^\beta|}{|P(\xi, \zeta_0/|\zeta_0|)|}\, d\xi \leqslant c_2 \left|P\left(0, \operatorname{Im}\frac{\zeta_0}{|\zeta_0|}\right)\right|^{-1}.$$

But

$$\operatorname{Im}\frac{\zeta_0}{|\zeta_0|} = (1+|\sigma|^2)^{-1/2} n_0,$$

therefore

$$\int\limits_{|\xi|\leqslant B} \frac{|\xi^\beta|}{|P(\xi, \zeta_0/|\zeta_0|)|}\, d\xi \leqslant c_3(1+|\sigma|^2)^{m/2},$$

and so (5.2) holds.

PROPOSITION 8. *Suppose* $|\alpha|+|\gamma| < m-\nu$ *and* $c_\alpha \in H_p^+$ *with*

$$p > |\alpha|+|\gamma|+\frac{1}{2}(n+1+\nu). \tag{5.3}$$

Then the integral formula for the inverse Fourier transform on the right-hand side of (5.1) *holds.*

Proof. It is sufficient to prove the convergence of the integral

$$\int\limits_{R^{n+1-\nu}} |\hat{c}_\alpha(\zeta_0)| G_{\alpha+\gamma}(\zeta_0)\, d\sigma;$$

this can be done by using (5.2) and the Schwarz inequality in $L_2(R^{n+1-\nu})$.

COROLLARY 2. *Suppose that* $c_\alpha \in H_p^+$ *with* p *satisfying* (5.3). *Then for* $|\gamma| \leqslant m_\beta$ *and* $|\alpha| < l_k$ *we have*

$$D_x^\gamma u_\alpha|_{x=0} = (2\pi)^{-n-1} \int\limits_{R^{n+1-\nu}} [\exp(i(s, \zeta_0))]\, \hat{c}_\alpha(\zeta_0) I_{\alpha+\gamma}(\zeta_0)\, d\sigma, \tag{5.4}$$

where

$$I_\beta(\zeta) = \int_{R^\nu} \frac{\xi^\beta}{P(\xi, \zeta)}\, d\xi \qquad (\zeta \in R^{n+1-\nu} - i\Gamma_0).$$

6. Reduction of the boundary value problem to a system of convolution equations on the boundary. Now we shall construct the linear system of convolution equations from which the unknown coefficients c_a can be computed.

PROPOSITION 9. *If* $d_\beta < 0$, *then*

(i) $I_\beta(\zeta)$ *is a holomorphic function in* $R^{n+1-\nu} - i\Gamma_0$;

(ii) I_β *is homogeneous of degree* d_β;

(iii) *for every closed cone* $L \subset \Gamma_0$ *there exists a positive constant* c *such that*

$$|I_\beta(\zeta)| \leqslant c|\mathrm{Im}\,\zeta|^{-m} \tag{6.1}$$

for $\zeta \in R^{n+1-\nu} - iL$, $|\zeta| = 1$, $\mathrm{Im}\,\zeta \neq 0$.

Proof. The same reasoning as in the proof of Proposition 4 shows that I_β is of class C^1 with respect to the real variables σ_j and n_k. Moreover, the derivatives D_{σ_j} and D_{n_k} can be taken under the sign of the integral. As the integrand is obviously a holomorphic function of ζ, it satisfies the Cauchy-Riemann equations, and so does I_β. This gives (i).

Statement (ii) can be verified immediately; it remains to prove (iii). Using Proposition 3 (with K being the unit sphere) we have to estimate two integrals, namely $\int_{|\xi|\leqslant B}$ and $\int_{|\xi|\geqslant B}$. The second one converges uniformly with respect to ζ and may be estimated by a constant. For the first one we have from (1.2)

$$\left|\int_{|\xi|\leqslant B} \frac{\xi^\beta}{P(\xi, \zeta)}\, d\xi\right| \leqslant c|P(0, \mathrm{Im}\,\zeta)|^{-1}$$

(with c depending on B) and, by the homogeneity of P,

$$|P(0, \mathrm{Im}\,\zeta)|^{-1} \leqslant c_L^{-1}\, |\mathrm{Im}\,\zeta|^{-m}, \qquad \text{where } c_L = \inf_{\substack{|n|=1\\ n\in L}} |P(0, n)|,$$

which yields (iii).

According to (4.4) we have $d_{a+\gamma} < 0$ for a, γ under consideration. Thus in virtue of Proposition 9 and Theorem I formula (5.4) can be rewritten as

$$D_x^\gamma u_a|_{x=0} = (2\pi)^{-\nu} e^t F^{-1}[\hat{c}_a(\zeta_0) E_{a+\gamma}(\zeta_0)], \tag{6.2}$$

where E_β is a tempered distribution with $\hat{E}_\beta(\zeta) = I_\beta(\zeta)$ $(\zeta \in R^{n+1-\nu} - i\Gamma_0)$ and $\operatorname{supp} E_\beta \subset \Gamma_0^*$. But, in view of Propositions 1-3, we have

$$\hat{c}_\alpha \hat{E}_{\alpha+\gamma} = F[e^{-t}(c_\alpha * E_{\alpha+\gamma})],$$

so (6.2) takes the form

$$D_x^\alpha u_\alpha|_{x=0} = (2\pi)^{-\nu} c_\alpha * E_{\alpha+\gamma}, \tag{6.3}$$

and the boundary conditions (4.3) yield the linear system of convolution equations

$$\sum_{|\alpha|<l_k} d_{\beta\alpha} * c_\alpha = g_\beta \quad (|\beta| < l_k), \tag{Λ}$$

where

$$d_{\beta\alpha}(s) = (2\pi)^{-\nu} \sum_{|\gamma|\leqslant m_\beta} b_{\beta\gamma}(D_s) E_{\alpha+\gamma}(s) \tag{6.4}$$

and

$$B_\beta = \sum_{|\gamma|\leqslant m_\beta} b_{\beta\gamma}(D_s) D_x^\gamma.$$

THEOREM 3. *Suppose that* $c_\alpha \in H^+_{k+r+|\alpha|+\nu/2}$ *is a solution of* (Λ) *and that*

$$r > \frac{n+1-\nu}{2}. \tag{6.5}$$

Then (4.8) *gives the solution of* $(\pi_{k,r})$ *and the following estimate holds:*

$$|u_\alpha|_{k,r} \leqslant c|c_\alpha|_{k+r+|\alpha|+\nu/2}.$$

If, moreover,

$$r > \frac{n+1-\nu}{2} + m, \tag{6.6}$$

then, conversely, each solution of $(\pi_{k,r})$ *is of the form* (4.8), *where the distributions* $c_\alpha \in H^+_{k+r+|\alpha|+\nu/2-m}$ *satisfy* (Λ).

In this sense the boundary value problem $(\pi_{k,r})$ *and the system of convolution equations* (Λ) *are equivalent.*

Proof. It follows from (6.5) that the assumptions of Corollary 2 are satisfied and, therefore, (Λ) is equivalent to (4.3). To prove the first part of the theorem we have to show that u given by (4.8) belongs to $H^+_{k,r}$. It follows from the well-known properties of the convolution product that $\operatorname{supp} u_\alpha$ is contained in the half-space $\{t \geqslant 0\}$ for each α, and so is u. It remains to prove that

$$(6.7) \qquad e^{-t}u_\alpha \in H_{k,r} \quad (|\alpha| < l_k)$$

or, equivalently, that the function

$$w_{\alpha,k,r}(\xi, \sigma) = (|\xi|^2 + |\zeta_0|^2)^{k/2} |\zeta_0|^r \hat{u}_\alpha(\zeta_0)$$

belongs to $L_2(R^{n+1})$. By (5.1) we have

(6.8)

$$\|w_{\alpha,k,r}\|^2_{L_2(R^{n+2})} = \int\limits_{R^{n+1-\nu}} |\zeta_0|^{2r} |c_\alpha(\zeta_0)|^2 \int\limits_{R^\nu} (|\xi|^2 + |\zeta_0|^2)^k \frac{\xi^{2\alpha}}{|P(\xi, \zeta_0)|^2} \, d\xi \, d\sigma.$$

As the integrand is a non-negative function, it suffices to prove the convergence of the iterated integral. Substituting $\xi = |\zeta_0| \varrho$ we can rewrite (6.8) as

$$(6.9) \qquad \|w_{\alpha,k,r}\|^2_{L_2(R^{n+1})} = \int\limits_{R^{n+1-\nu}} |\zeta_0|^{2(k+r+|\alpha|-m)+\nu} |\hat{c}_\alpha(\zeta_0)|^2 F_{k,\alpha}\left(\frac{\zeta_0}{|\zeta_0|}\right) d\sigma,$$

where

$$F_{k,\alpha}(\zeta) = \int\limits_{R^\nu} (1 + |\varrho|^2)^k \frac{\varrho^{2\alpha}}{|P(\varrho, \zeta_0)|^2} \, d\varrho.$$

To estimate the function $F_{k,\alpha}$ we use Proposition 3 (with K being the unit sphere) and (1.2), obtaining

$$\left| F_{k,\alpha}\left(\frac{\zeta_0}{|\zeta_0|}\right) \right| \leqslant c(1 + |\sigma|^2)^m.$$

Thus the integrand on the right-hand side of (6.9) does not exceed the function $c(1 + |\sigma|^2)^{k+r+|\alpha|+\nu/2} |\hat{c}_\alpha(\zeta_0)|^2$ which is summable over $R^{n+1-\nu}$, according to our assumptions. Thus the first part of the theorem is proved. The second part follows from the preceding investigations if we remark that (6.6) yields the assumptions of Corollary 2.

7. Solvability of the system (Λ) and of the problem ($\pi_{k,r}$). In virtue of Theorem 3 the boundary value problem under consideration is reduced to the system of convolution equations (Λ). To study its solvability let us introduce the matrix $\hat{D}(\zeta) \overset{\text{df}}{=} [\hat{d}_{\beta\alpha}(\zeta)]$, where

$$(7.1) \qquad \hat{d}_{\beta\alpha}(\zeta) = (2\pi)^{-\nu} \sum_{|\gamma| \leqslant m_\beta} b_{\beta\gamma}(\zeta) I_{\alpha+\gamma}(\zeta) \qquad (\zeta \in R^{n+1-\nu} - i\Gamma_0)$$

is the Fourier transform of (6.4). Our further considerations will be based on the following assumption:

(A_θ) There exists an open convex cone $\tilde{\Gamma}_0 \subset \Gamma_0$ with the following properties:

(i) $n_0 \in \tilde{\Gamma}_0$;

(ii) for every closed convex cone $L \subset \tilde{\Gamma}_0$ there are two numbers $c > 0$ and $\theta \geqslant 0$ such that

$$|\det \hat{D}(\zeta)| \geqslant c |\mathrm{Im}\,\zeta|^\theta \tag{7.2}$$

for $\zeta \in R^{n+1-\nu} - iL$, $|\zeta| = 1$, $\mathrm{Im}\,\zeta \neq 0$.

It follows from (7.2) that the matrix $\hat{D}(\zeta)$ is non-singular in $R^{n+1-\nu} - i\tilde{\Gamma}_0$. Let $R(\zeta) = [r_{\alpha\beta}(\zeta)]$ be its inverse.

PROPOSITION 10. *Assumption* (A_θ) *implies the following statements*:

(i) *the functions* $r_{\alpha\beta}$ *are holomorphic in* $R^{n+1-\nu} - i\tilde{\Gamma}_0$;

(ii) $r_{\alpha\beta}$ *is homogeneous of degree* $m - m_\beta - |\alpha| - \nu$;

(iii) *if* s_k *is the dimension of the matrix* $\hat{D}(\zeta)$, *then*

$$|r_{\alpha\beta}(\zeta)| \leqslant c |\mathrm{Im}\,\zeta|^{-(s_k-1)m-\theta} \tag{7.3}$$

for $\zeta \in R^{n+1-\nu} - iL$, $|\zeta| = 1$, $\mathrm{Im}\,\zeta \neq 0$.

Proof. As is known,

$$r_{\alpha\beta}(\zeta) = \frac{\hat{D}_{\beta\alpha}(\zeta)}{\det \hat{D}(\zeta)}, \tag{7.4}$$

where $\hat{D}_{\beta\alpha}(\zeta)$ is the algebraic complement of the element $\hat{d}_{\beta\alpha}(\zeta)$. Thus (i) follows from (7.1), (7.2), and Proposition 9. Let us note further that the polynomials $b_{\beta\gamma}$ are homogeneous of degree $m_\beta - |\gamma|$; therefore by (7.1) and Proposition 9 the function $\hat{d}_{\beta\alpha}$ is homogeneous of degree $m_\beta + |\alpha| - m + \nu$. So (ii) follows from (7.4) by elementary reasoning. Estimate (7.3) follows easily from (6.1) and (7.2). The proof is completed.

Let now θ_0 be a non-negative number such that (7.2) is valid with $\theta = \theta_0$ for the cone L consisting of all vectors of the form λn_0 ($\lambda \geqslant 0$). Putting for the sake of simplicity

$$p = k + r - \nu/2 + ms_k + \theta_0,$$

we can prove now our main result:

THEOREM 4. *Suppose that assumptions* (A_1)-(A_3), (A_θ) *hold and that* $r > (n+1-\nu)/2$. *Then for an arbitrary* $g_\beta \in H^+_{p-m_\beta}$ ($|\beta| < l_k$) *formula* (4.8) *with* c_α *computed from* (Λ) *gives the solution* u *of* ($\pi_{k,r}$) *satisfying the energy inequality*

$$|u|_{k,r} \leqslant c \sum_{|\beta| < l_k} |g_\beta|_{p-m_\beta}. \tag{7.5}$$

If, moreover, $r > (n+1-\nu)/2 + m$, *then this solution is unique.*

The proof will be based on the following

LEMMA. *Let $F_{\alpha,\beta,n}(\sigma) = r_{\alpha\beta}(\sigma - in)$ for a fixed $n \in \tilde{\Gamma}_0$. Then $F_{\alpha,\beta,n} \in O_M$.*

This lemma may be easily derived from (7.2) and Proposition 4 by using the homogeneity of $\hat{D}(\zeta)$.

Proof of Theorem 4. Passing to the Fourier transforms we can write the system (Λ) in the equivalent form

$$(\hat{\Lambda}) \qquad \sum_{|\alpha|<l_k} \hat{d}_{\beta\alpha}(\zeta_0)\hat{c}_\alpha(\zeta_0) = \hat{g}_\beta(\zeta_0),$$

which yields

$$\hat{c}_\alpha(\zeta_0) = \sum_{|\beta|<l_k} r_{\alpha\beta}(\zeta_0)\hat{g}_\beta(\zeta_0). \tag{7.6}$$

It can be immediately proved, by using Propositions 1 and 2, that c_α defined by (7.6) satisfy (Λ).

By Proposition 10 and Theorem I, $r_{\alpha\beta}(\zeta) = \hat{q}_{\alpha\beta}(\zeta)$ $(\zeta \in R^{n+1-\nu} - i\tilde{\Gamma}_0)$, where $q_{\alpha\beta}$ is a tempered distribution with support contained in $\tilde{\Gamma}_0^*$. In virtue of Propositions 1, 2 and Lemma 4, identity (7.6) can be rewritten in the form

$$c_\alpha = \sum_{|\beta|<l_k} q_{\alpha\beta} * g_\beta.$$

It is now evident that $\operatorname{supp} c_\alpha$ is contained in the half-space $\{t \geqslant 0\}$ if all the g_β's are. Noting that

$$|\zeta_0| = (1+|\sigma|^2)^{1/2} \quad \text{and} \quad \left|\operatorname{Im} \frac{\zeta_0}{|\zeta_0|}\right| = (1-|\sigma|^2)^{-1/2},$$

from Proposition 10 we get

$$|r_{\alpha\beta}(\zeta_0)| \leqslant c(1+|\sigma|^2)^{(m-m_\beta-|\alpha|-\nu+\theta_1)/2}, \quad \text{where} \quad \theta_1 = (s_k-1)m+\theta_0.$$

Thus (7.6) and the above-assumed regularity of the boundary data g_β assure that c_α have the regularity required in Theorem 3, and so our statement holds.

8. An example. We give here a simple example of a boundary value problem satisfying conditions (A_1)-(A_3), (A_θ). Let us put $\nu = 2$, $n = 3$, and

$$P(\xi, \eta, \tau) = (|\xi|^2 + \eta^2 - \tau^2)^2.$$

For $k \in [2, 3)$, (4.7) is obviously satisfied and $l_k \in (0, 1]$. Thus we have only one boundary condition in the problem $(\pi_{k,r})$. Suppose that it is of Dirichlet type, so $m_\beta = 0$ and (4.4) holds. Conditions (A_1)-(A_3) are evidently satisfied; it remains to prove that (A_θ) holds. Thus we have to estimate from below the function

$$I(\zeta) = \int \frac{d\xi}{P(\xi, \zeta)}$$

(for the sake of simplicity we omit the index 0). Obviously, it is sufficient to estimate one of the expressions $|\mathrm{Re}\, I(\zeta)|$ or $|\mathrm{Im}\, I(\zeta)|$. Simple calculations show that

$$\mathrm{Re}\, I(\zeta) = \int \frac{(|\xi|^2+w)^2-4c^2}{[(|\xi|^2+w)^2+4c^2]^2}\, d\xi \tag{8.1}$$

and

$$\mathrm{Im}\, I(\zeta) = -4c \int \frac{|\xi|^2+w}{[(|\xi|^2+w)^2+4c^2]^2}\, d\xi, \tag{8.2}$$

where $w = \eta^2-\tau^2-a^2+b^2$, $c = \tau b-\eta a$, $\mathrm{Re}\,\zeta = (\eta, \tau)$, $\mathrm{Im}\,\zeta = -(a, b)$.

The cone Γ_0 is defined by

$$a^2 < b^2, \quad b > 0. \tag{8.3}$$

We also have

$$|\zeta|^2 = \eta^2+\tau^2+a^2+b^2 = 1, \tag{8.4}$$

which yields

$$|w| \leqslant 1. \tag{8.5}$$

Evidently, we can restrict our further considerations to the case $\tau \geqslant 0$ (if $\tau < 0$, then replacing (η, τ) by $(-\eta, -\tau)$ we change only the sign of $\mathrm{Im}\, I(\zeta)$). Let us suppose now that (a, b) belongs to a closed cone $L \subset \Gamma_0$ and consider first the case where

$$\eta^2+\tau^2 \leqslant d^2. \tag{8.6}$$

Consequently, by (8.4),

$$a^2+b^2 \geqslant 1-d^2. \tag{8.7}$$

We have

$$\mathrm{Re}\, I(\zeta)\big|_{\eta=\tau=0} = \int \frac{d\xi}{(|\xi|^2+w)^2},$$

whence, by (8.5),

$$\mathrm{Re}\, I(\zeta)\big|_{\eta=\tau=0} \geqslant \int \frac{d\xi}{(|\xi|^2+1)^2} = 2\varkappa > 0.$$

For an arbitrary fixed $d < 1$ the integral $\mathrm{Re}\, I(\zeta)$ is a uniformly continuous function of real variables η, τ, a, b. Therefore, there exists a $d_0 \in (0, 1)$ such that for η, τ satisfying (8.6) with $d = d_0$ we have

$$\mathrm{Re}\, I(\zeta) > \varkappa. \tag{8.8}$$

Suppose now that

$$\eta^2 + \tau^2 > d_0^2 \tag{8.9}$$

and let θ be the angle between the vectors (a, b) and $(-\eta, \tau)$. There exists a closed cone Γ_L, symmetric with respect to the τ-axis, having the following properties:

(i) $\Gamma_0 \subset \Gamma_L$;

(ii) there is a positive number d_1 such that

$$|\cos\theta| \geqslant d_1 \tag{8.10}$$

for $(\eta, \tau) \in \Gamma_L$ and $(a, b) \in L$.

Now

$$|c| = \sqrt{a^2 + b^2}\sqrt{\eta^2 + \tau^2}\cos\theta,$$

and therefore (8.9) and (8.10) yield

$$|c| \geqslant d_0 d_1 |\mathrm{Im}\,\zeta| \tag{8.11}$$

for $(\eta, \tau) \in \Gamma_L$. Introducing the polar coordinates we get

$$\mathrm{Im}\, I(\zeta) = -8\pi c \int_0^\infty \frac{(r^2 + w) r}{[(r^2 + w)^2 + 4c^2]^2}\, dr$$

or, by the substitution $r^2 + w = \varrho$,

$$\mathrm{Im}\, I(\zeta) = -4\pi c \int_{|w|}^\infty \frac{\varrho\, d\varrho}{(\varrho^2 + 4c^2)^2}. \tag{8.12}$$

Using (8.5), (8.11), and the obvious inequality

$$|c| \leqslant 1, \tag{8.13}$$

we get

$$\int_{|w|}^\infty \frac{\varrho\, d\varrho}{(\varrho^2 + 4c^2)^2} \geqslant \frac{1}{10}, \tag{8.14}$$

and therefore from (8.11) and (8.12) we obtain

$$|\mathrm{Im}\, I(\zeta)| \geqslant \frac{2\pi}{5} d_0 d_1 |\mathrm{Im}\,\zeta \tag{8.15}$$

for $(\eta, \tau) \in \Gamma_L$ satisfying (8.9).

It remains to prove this estimate for (η, τ) satisfying (8.9) but lying outside of Γ_L. In this case there is a positive constant $\varkappa_1$ such that $\eta^2 - \tau^2 \geqslant \varkappa_1$, and therefore

$$w \geqslant \varkappa_1. \tag{8.16}$$

It can be easily proved that the integral $\operatorname{Re} I(\zeta)$ is a uniformly continuous function of the parameters c, w in the set described by (8.5), (8.13), and (8.16).

Moreover, we have

$$\operatorname{Re} I(\zeta)|_{c=0} \geqslant \int \frac{d\xi}{(|\xi|^2+1)^2} = 2\varkappa > 0,$$

and therefore there exists a positive constant d_3 such that

$$\operatorname{Re} I(\zeta) \geqslant \varkappa \tag{8.17}$$

for $|c| < d_3$. If $|c| \geqslant d_3$, then (8.12) and (8.14) yield

$$|\operatorname{Im} I(\zeta)| \geqslant \frac{2\pi}{5} d_3. \tag{8.18}$$

It follows from (8.8), (8.15), (8.17), and (8.18) that (A_θ) holds with $\theta = 1$ (not depending on the cone L). The energy inequality (7.5) connected with the problem $(\pi_{k,r})$ considered in this section takes the form

$$|u|_{k,r} \leqslant c\,|g|_{k+r+4}.$$

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