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QUASI-PROJECTIVE AND QUASI-INJECTIVE FINITE VALUATED GROUPS

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Introduction. The notion of valuated vector space was introduced by Fuchs [2], and generalized to the notion of valuated group by Richman and Walker [4]. In the latter paper, the projective and injective valuated groups are characterized. In this note, we characterize the quasi-projective and quasi-injective finite valuated groups.

We begin with the necessary definitions. A *valuated group* G is an abelian group with a valuation $v_p: G \to \text{ordinals} \cup \{\infty\}$ for each prime p satisfying:

(1) $v_p(x+y) \geqslant \min\{v_p(x), v_p(y)\}$,

(2) $v_p(px) > v_p(x)$,

(3) $v_p(cx) = v_p(x)$ if $(c, p) = 1$.

The valuated group G is called *p-local* if $v_q(g) = \infty$ for all $g \in G$, $q \neq p$. In this case we write $v_p(g) = v(g)$. Note that a valuated p-group is p-local.

In what follows, we work in $\mathscr{F}$, the category of finite valuated groups. It is clearly sufficient to work in $\mathscr{F}_p$, the category of all finite valuated p-groups. The maps in this category are the valuated homomorphisms, i.e. the abelian group homomorphisms f such that $v(x) \leqslant v(f(x))$. Unless otherwise specified, $\bigoplus_{i=1}^{n} G_i$ denotes a direct sum in $\mathscr{F}_p$, the ordinary abelian direct sum with

$$v(g_1 \oplus g_2 \oplus \ldots \oplus g_n) = \min\{v(g_i)\}.$$

If $G \in \mathscr{F}_p$, and H is a subgroup of G, then H inherits the valuation from G, and G/H is assigned (see [4]) the valuation

$$v(g+H) = \max\{v(g+h) \mid h \in H\}.$$

Abelian group notation follows Fuchs [1]. Valuated group notation follows Richman and Walker [4]. In particular, if $G \in \mathscr{F}_p$,

$$\operatorname{supp} G = \{a \mid a = v(g) \text{ for some } g \in G\}$$

and

$$G^{\gamma} = \{g \in G \mid v(g) \geqslant \gamma\} \quad \text{for any ordinal } \gamma.$$

Finally, we employ the usual definitions for quasi-projective and quasi-injective. A group $G \in \mathscr{F}_p$ is *quasi-projective* if for any subgroup $H \subset G$ and any map $f: G \to G/H$ there is a map $\bar{f}: G \to G$ which lifts f. Dually, G is *quasi-injective* if for any subgroup H and any map $f: H \to G$ there is a map $\bar{f}: G \to G$ which lifts f.

1. Quasi-projectives in $\mathscr{F}_p$. We show first that every quasi-projective in $\mathscr{F}_p$ is a direct sum of cyclics.

THEOREM 1.1. *If G is quasi-projective in $\mathscr{F}_p$, then G is a direct sum of cyclics.*

Proof. (By induction on the minimal k such that $p^kG = 0$.) If $k = 1$, then the result is immediate by [2], Theorem 1, or [3], Theorem 4.

Assume that $k > 1$ and consider $\bar{G} = G/G[p] \cap pG$. Since $G[p] \cap pG$ is fully invariant in G, $\bar{G}$ is quasi-projective, hence, by induction, is a direct sum of cyclics. Write

$$\bar{G} = \bigoplus_{i=1}^{n} \langle \bar{x}_i \rangle, \qquad \text{where } \bar{x}_i = x_i + G[p] \cap pG.$$

Now choose a minimal γ so that $(G[p])^{\gamma} \neq G[p]$. Then, for some $i = i_0$,

$$\operatorname{socle}\langle x_{i_0}\rangle \nsubseteq (G[p])^{\gamma}$$

(this is true since $\{x_i\}$ generates G). Let π_0 be the projection of $\bar{G}$ onto $\langle \bar{x}_{i_0}\rangle$, and f the quasi-projectively induced lifting of $G \to \bar{G} \xrightarrow{\pi_0} \langle \bar{x}_{i_0}\rangle$. Then $f(G) \subseteq \langle x_{i_0}\rangle + G[p]$. However, by Lemma 3 of [2], we may write

$$G[p] = \operatorname{socle}\langle x_{i_0}\rangle \oplus A \oplus (G[p])^{\gamma},$$

where A is homogeneous of value $\gamma' < \gamma$. Thus

$$\langle x_{i_0}\rangle + G[p] = \langle x_{i_0}\rangle \oplus A \oplus (G[p])^{\gamma},$$

a direct sum in $\mathscr{F}_p$. So we may follow f by projection onto $\langle x_{i_0}\rangle$ to obtain a valuated map $f': G \to \langle x_{i_0}\rangle$. It is easy to see that $f'(\langle x_{i_0}\rangle) = \langle x_{i_0}\rangle$ so that f' will yield a splitting $G = \langle x_{i_0}\rangle \oplus \operatorname{Ker} f'$. But now $\operatorname{Ker} f'$, as a summand of G, is quasi-projective and the argument may be repeated to obtain G as a direct sum of cyclics.

THEOREM 1.2. *Let*

$$G = \bigoplus_{i=1}^{n} \langle x_i \rangle \ \textit{in } \mathscr{F}_p \quad \textit{with } \operatorname{order} x_i = p^{k_i}.$$

Then G is quasi-projective if and only if, for all $m \geqslant 0$, (QP1) *if* $\text{order}\, x_i < \text{order}\, p^m x_j$, *then* $v(x_i) > v(p^m x_j)$;

(QP2) *if* $v(x_i) \leqslant v(p^m x_j)$, *then* $v(p^t x_i) \leqslant v(p^{t+m} x_j)$ *for all* t, $1 \leqslant t \leqslant k_i - 1$.

Proof. Necessity. Suppose that G is quasi-projective. Let $\text{order}\, x_i < \text{order}\, p^m x_j$ and assume that $v(x_i) \leqslant v(p^m x_j)$. Then there is a valuated map

$$f: \frac{\langle x_i \rangle}{\langle p x_i \rangle} \to \frac{\langle x_j \rangle}{\langle p^{m+1} x_j \rangle}$$

given by

$$x_i + \langle p x_i \rangle \xrightarrow{f} p^m x_j + \langle p^{m+1} x_j \rangle.$$

As a summand of G, $\langle x_i \rangle \oplus \langle x_j \rangle$ is quasi-projective. Hence

$$\langle x_i \rangle \oplus \langle x_j \rangle \to \frac{\langle x_i \rangle}{\langle p x_i \rangle} \oplus \frac{\langle x_j \rangle}{\langle p^{m+1} x_j \rangle} \xrightarrow{f \oplus 0} \frac{\langle x_i \rangle}{\langle p x_i \rangle} \oplus \frac{\langle x_j \rangle}{\langle p^{m+1} x_j \rangle}$$

can be lifted. Since $\text{order}\, x_i < \text{order}\, p^m x_j$, this is clearly impossible. This proves (QP1).

To prove (QP2) assume that $v(x_i) \leqslant v(p^m x_j)$ again. Arguing as above, the map f must lift to a valuated map $\bar{f}: \langle x_i \rangle \to \langle x_j \rangle$. It is easy to check that $\bar{f}(x_i) = c p^m x_j$, where $(c, p) = 1$. Thus, $v(p^t x_i) \leqslant v(p^{t+m} x_j)$ for all t, $1 \leqslant t \leqslant k_i - 1$.

Sufficiency. Suppose that G satisfies (QP1) and (QP2). Let $f: G \to G/K$ be valuated. For each i write $f(x_i) = u_i + K$, where $u_i \in G$ with $v(u_i) = v(u_i + K)$. Since $G = \oplus \langle x_i \rangle$ in $\mathscr{F}_p$, it is enough to show that $\text{order}\, u_i \leqslant \text{order}\, x_i$ and $v(p^r x_i) \leqslant v(p^r u_i)$, $1 \leqslant r \leqslant k_i - 1$. Indeed, $x_i \to u_i$ induces then a valuated endomorphism of G, which clearly lifts f. Say

$$u_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Now $v(x_i) \leqslant v(u_i + K) = v(u_i) \leqslant v(a_{ij} x_j)$ for all j. By (QP2), $v(p^r x_i) \leqslant v(p^r a_{ij} x_j)$ for all j, and hence $v(p^r x_i) \leqslant v(p^r u_i)$, $1 \leqslant r \leqslant k_i - 1$. If $\text{order}\, u_i > \text{order}\, x_i$, then $\text{order}\, a_{ij} x_j > \text{order}\, x_i$ for some j. Thus, by (QP1), $v(x_i) > v(a_{ij} x_j) \geqslant v(u_i)$ — a contradiction.

2. Quasi-injectives in $\mathscr{F}_p$. In the quasi-injective case we are able to prove a stronger version of Theorem 1.1.

We begin with a useful lemma.

LEMMA 2.1. *Let G be a valuated group, and $\bigoplus_i \langle x_i \rangle \oplus C$ a valuated direct sum contained in G such that $v(x_i) \leqslant v(c)$ for all x_i and $c \in C$. Suppose that for each i there is $a_i \in G$ with $p a_i = x_i$. Then $A \oplus C$ is a valuated direct sum in G, where $A = \bigoplus_i \langle a_i \rangle$ as an abelian group.*

Proof. It is easy to check that $\bigoplus_i \langle a_i \rangle \oplus C$ is an abelian group direct sum. Since $v(a_i) < v(c)$ for all a_i and $c \in C$, it follows that projection onto A is a valuated map.

THEOREM 2.1. *Let G be a countable bounded quasi-injective valuated group whose support is order isomorphic to a subset of the set Z^+ of positive integers. Then G is a direct sum of cyclics as a valuated group.*

Proof. (By induction on k, where $p^k G = 0$.) If $k = 1$, then the result follows from Theorem 1 in [2].

For $k > 1$, consider $G' = G[p] + pG$. As a fully invariant subgroup of G, G' is quasi-injective. Hence, by induction,

$$G' = \bigoplus_i \langle y_i \rangle \bigoplus_j \langle x_j \rangle,$$

where the y_i's and x_j's are chosen so that $h_p^G(y_i) = 0$ and $h_p^G(x_j) \geqslant 1$ for all i, j. Let

$$Y = \bigoplus_i \langle y_i \rangle$$

and let π_Y be the natural projection of G' onto Y. By quasi-injectivity of G, π_Y lifts to a valuated endomorphism $f: G \to G$. Note that $\operatorname{Im} f \subseteq Y + G[p] \subseteq G'$. Therefore, we may follow f by π_Y to get a splitting $G = Y \oplus G_1$, where $G_1 = \operatorname{Ker} \pi_Y f$. Since $pY = 0$, Y is a direct sum of cyclics (by [2], Theorem 1).

As a summand of G, G_1 is also quasi-injective. Furthermore,

$$pG_1 = \bigoplus_j \langle x_j \rangle$$

as a valuated group. Write $\{v(x_j)\}$ as $\{\gamma_1 < \gamma_2 < \gamma_3 < \ldots\}$ and let

$$X_1 = \bigoplus_{j \in J_1} \langle x_j \rangle, \qquad \text{where } J_1 = \{j \mid v(x_j) = \gamma_1\}.$$

For all $j \in J_1$, choose $a_j \in G_1$ such that $pa_j = x_j$, and let $A_1 = \bigoplus_{j \in J_1} \langle a_j \rangle$ (as an abelian group). Now let π_1 be the projection of pG_1 onto X_1. By quasi-injectivity, π_1 lifts to $f_1: G_1 \to G_1$. Note that

$$\operatorname{Im} f_1 \subseteq A_1 + G_1[p] \subseteq A_1 + \bigoplus_{j \notin J_1} \langle x_j \rangle.$$

By Lemma 2.1, $A_1 \oplus \bigoplus_{j \notin J_1} \langle x_j \rangle$ is a valuated direct sum. Hence we may follow f by projection onto A_1 to obtain a splitting

$$G_1 = A_1 \oplus G_2, \qquad \text{where } \bigoplus_{j \notin J_1} \langle x_j \rangle \subseteq G_2.$$

Continuing as above, we obtain a valuated direct sum

$$G_1 = \bigoplus_{i=1} A_i \qquad \text{with } pA_i = \bigoplus_{j \in J_i} \langle x_j \rangle \text{ and } J_i = \{j \mid v(x_j) = \gamma_i\}.$$

Each A_i is quasi-injective and the technique above can be used to obtain A_i as a direct sum of cyclics (possibly countable). Thus G is a valuated direct sum of cyclics.

COROLLARY 2.1. *Any quasi-injective in $\mathscr{F}_p$ is a direct sum of cyclics.*

THEOREM 2.2. *Let*

$$G = \bigoplus_{i=1}^{n} \langle x_i \rangle$$

be a direct sum of cyclics in $\mathscr{F}_p$, with $\operatorname{order} x_i = p^{k_i}$, $1 \leqslant i \leqslant n$. *Then G is quasi-injective if and only if*

(QI1) $k_i < k_j$ *implies* $v(p^{k_i-1}x_i) < v(p^{k_i}x_j)$;

(QI2) *if* $v(p^t x_i) \leqslant v(p^{k_j-1}x_j)$ *for some* $t \geqslant 0$, *then* $v(p^{t-s}x_i) \leqslant v(p^{k_j-1-s}x_j)$ *for all* s, $0 \leqslant s \leqslant t$.

Proof. Necessity. Assume that G is quasi-injective. If $k_i < k_j$, but $v(p^{k_i-1}x_i) \geqslant v(p^{k_i}x_j)$, then $f\colon \langle p^{k_i}x_j \rangle \to G$, given by $f(p^{k_i}x_j) = p^{k_i-1}x_i$, is a valuated map which does not lift to an endomorphism of G (since it would decrease height). This proves (QI1).

Now suppose that $v(p^t x_i) \leqslant v(p^{k_j-1}x_j)$. Then $f(p^t x_i) = p^{k_j-1}x_j$ defines a valuated map $f\colon \langle p^t x_i \rangle \to G$, which must lift to a map $\langle x_i \rangle \to \langle x_j \rangle$ in which $x_i \to cp^{k_j-t-1}x_j$ with $(c, p) = 1$. This implies (QI2).

Sufficiency. Assume that G satisfies (QI1) and (QI2).

Let H be a subgroup of G and let $f\colon H \to G$ be a valuated map. We will show that f can be lifted by induction on k, the number of non-zero projections $\pi_i(H)$, where π_i is the projection of G onto $\langle x_i \rangle$, $1 \leqslant i \leqslant n$. If $k = 1$, f can be lifted directly by (QI2).

For $k > 1$, we consider two cases.

Case I. Suppose that $H \cap \langle x_i \rangle \neq (0)$ for some i. Then, using (QI2), it is easy to lift

$$f|_{H\cap\langle x_i\rangle} \oplus 0\colon H \cap \langle x_i \rangle \oplus [\bigoplus_{j\neq i} \langle x_j \rangle] \to G$$

to a map $f_i\colon G \to G$. Note that $f_i(1-\pi_i) = 0$. Then $f - f_i$ induces a map $f'\colon H' \to G$, where $H' = (1-\pi_i)H$, by

$$f'(1-\pi_i)(h) \equiv (f-f_i)(h).$$

This map is well defined since if $(1-\pi_i)(h_1) = (1-\pi_i)h_2$, then $h_1 - h_2 \in H \cap \langle x_i \rangle$, so $(f-f_i)(h_1-h_2) = 0$. By induction, since H' has at least one non-zero projection less than H, f' can be lifted to a map $\bar{f}\colon G \to G$ which can be chosen with $\bar{f}(x_i) = 0$. It is easy to check that $\bar{f}(1-\pi_i) + f_i$ is the desired lifting of f.

Case II. Suppose that $H \cap \langle x_i \rangle = (0)$ for all i. In this case, choose i so that $v\big((\pi_i H)[p]\big)$ is maximal. As before, it is easy to check that $f'(1-\pi_i)(h) = f(h)$ defines a map from $(1-\pi_i)H$ to G. To prove that f'

is a valuated map it is enough to show that we cannot have $v(c_i x_i) < v(c_j x_j)$ for all $j \neq i$ for some element $(c_i x_i + \sum_{j \neq i} c_j x_j) \in H$. (Recall that G has the direct sum valuation.) Now, if the above were true for some element of H, choose p^k such that $0 \neq p^k c_i x_i \in \langle x_i \rangle [p]$. But

$$(*) \qquad v(c_j p^k x_j) > v(c_i p^k x_i) \quad \text{for all } j \neq i,$$

for otherwise, by (QI2), $v(c_j x_j) \leqslant v(c_i x_i)$ for some j. Thus $(*)$ contradicts the maximality of $v\big((\pi_i H)[p]\big)$ and f' is valuated. By the induction hypothesis, lift f' to $\bar{f}': G \to G$ and let $\bar{f} = \bar{f}'(1-\pi_i)$. Then $\bar{f}$ is the desired lifting of f.

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