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PRESERVING SOME PROPERTIES OF LARGE CARDINALS UNDER MILD COHEN EXTENSIONS

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In this paper we prove three theorems on preserving some large cardinals (weakly compact cardinals, Ramsey cardinals and Rowbottom cardinals) under mild Cohen extensions.

THEOREM 1. *If $\mathfrak{M}$ is a countable model of* ZFC, *$\varkappa$ is a weakly compact cardinal in $\mathfrak{M}$, $|\mathscr{C}| < \varkappa$, and $\mathscr{G}$ is $\mathscr{C}$-generic over $\mathfrak{M}$, then $\varkappa$ is weakly compact in $\mathfrak{M}[\mathscr{G}]$.*

THEOREM 2. *If $\mathfrak{M}$ is a countable model of* ZFC, *$\varkappa$ is a Ramsey cardinal in $\mathfrak{M}$, $|\mathscr{C}| < \varkappa$, and $\mathscr{G}$ is $\mathscr{C}$-generic over $\mathfrak{M}$, then $\varkappa$ is a Ramsey cardinal in $\mathfrak{M}[\mathscr{G}]$.*

THEOREM 3. *If $\mathfrak{M}$ is a countable model of* ZFC, *$\varkappa$ is a Rowbottom cardinal in $\mathfrak{M}$, and $\mathscr{C}$ satisfies* ccc, *then $\varkappa$ is a Rowbottom cardinal in $\mathfrak{M}[\mathscr{G}]$.*

Related problems were considered by Levy and Solovay [4], Jensen [3], Prikry [8], Paris [7] and McAloon [6]. We shall use the unramified forcing of Shoenfield (for details and notation see [11]).

Let $P(a)$ be a property of cardinals expressible in ZFC, and let $\varkappa$ be a cardinal such that $P(\varkappa)$ holds in a model $\mathfrak{M}$ of ZFC and the notion of forcing $\mathscr{C}$ is a set of power less than $\varkappa$. We say that *P is preserved under mild Cohen extension* if $P(\varkappa)$ holds in the Cohen extension. In the proofs of Theorems 2 and 3 we shall use the Boolean forcing of Mathias (for details see [5]).

Silver has proved in [12] that a cardinal $\varkappa$ is *weakly compact* iff the condition $\varkappa \to (\varkappa)^2$ is satisfied, i.e., iff

$$\forall A \subset [\varkappa]^2 \exists X \subset \varkappa, \quad |X| = \varkappa \wedge ([X]^2 \subseteq A \vee [X]^2 \subseteq [\varkappa]^2 - A).$$

Proof of Theorem 1. Let $A \subseteq [\varkappa]^2$ in $\mathfrak{M}[\mathscr{G}]$, and let a be a name for A ($K_{\mathscr{G}}(a) = A$). We write

$$d_p = \big\{\{a, \beta\} \in [\varkappa]^2 : p \Vdash \langle \hat{a}, \hat{\beta}\rangle^{\wedge} \in a\big\}.$$

Notice that if $p \in \mathscr{G}$, then

$$d_p \subseteq A \subseteq \bigcup_{q \leqslant p} d_q.$$

It is enough to show that there is a p such that $p \in \mathscr{G}$ and

$$\exists X \subset \varkappa \wedge |X| = \varkappa \wedge [X]^2 \subseteq d_p$$

or

$$\exists X \subset \varkappa \wedge |X| = \varkappa \wedge [X]^2 \subseteq [\varkappa]^2 - \bigcup_{q \leqslant p} d_q.$$

Let

$$\mathscr{D} = \{p\colon \exists X \in P^{\mathfrak{M}}(\varkappa) \wedge |X| = \varkappa \wedge [X]^2 \subseteq d_p \vee \exists X \in P^{\mathfrak{M}}(\varkappa) \wedge |X| = \varkappa \wedge [X]^2 \subseteq [\varkappa]^2 - \bigcup_{q \leqslant p} d_q\}.$$

We shall prove that $\mathscr{D}$ is $\mathscr{C}$-dense ($\forall p \in \mathscr{C} \exists q \in \mathscr{D}, q \leqslant p$).

Let $p \in \mathscr{C}$.

(i) If there is $X \in P^{\mathfrak{M}}(\varkappa)$, $|X| = \varkappa$, and

$$[X]^2 \subseteq [\varkappa]^2 - \bigcup_{q \leqslant p} d_q,$$

we can take $p = q$.

(ii) If (i) is not satisfied, then $|\{q \in \mathscr{G}, q \leqslant p\}| < \varkappa$ and there is a function g in $\mathfrak{M}$ such that g is 1-1 and $g\colon \{q \in \mathscr{C}\colon q \leqslant p\} \to \lambda - \{0\}$ for some $\lambda < \varkappa$.

But $\mathfrak{M} \models \varkappa \to (\varkappa)^2$, so $\mathfrak{M} \models \varkappa \to (\varkappa)^2_{<\varkappa}$ (see [12]), i.e.,

(1) $$\forall \lambda < \varkappa \; \forall f\colon [\varkappa]^2 \to \lambda \exists X \subset \varkappa \wedge |X| = \varkappa \wedge |f * [X]^2| = 1.$$

Let f be a function defined in the following way:

$$f(\{\alpha, \beta\}) = \begin{cases} g(q) & \text{if } q \leqslant p \text{ is a least condition (in some order) such that } \{\alpha, \beta\} \in d_q, \\ 0 & \text{if there is no such } q. \end{cases}$$

$f\colon [\varkappa]^2 - \lambda$, $f \in \mathfrak{M}$ and from (1) we have $\exists X \subset \varkappa$, $|X| = \varkappa |f * [X]^2| = 1$. Thus there is $a \neq 0$ such that $f * [X]^2 = \{a\}$ and we have a $q \leqslant p$ such that $[X]^2 \subseteq d_q$, for if $a = 0$, then

$$[X]^2 \subseteq [\varkappa]^2 - \bigcup_{q \leqslant p} d_q.$$

Since $\mathscr{G}$ is $\mathscr{C}$-generic and $\mathscr{D}$ is $\mathscr{C}$-dense, there are $p \in \mathscr{D} \cap \mathscr{G}$ and $X \subseteq \varkappa$ such that $|X| = \varkappa$ and $[X]^2 \subseteq A$ or $[X] \subseteq [\varkappa]^2 - A$.

We say that $\varkappa$ is a *Ramsey cardinal* iff $\varkappa \to (\varkappa)^{<\omega}$, where $\varkappa \to (\varkappa)^{<\omega}$ means that (see [12])

$$\forall f\colon [\varkappa]^{<\omega} \to 2, \quad \exists X \subseteq \varkappa, \quad |X| = \varkappa \forall n \, |f * [X]^n| = 1.$$

We replace a notion of forcing by a Boolean algebra $\boldsymbol{B}$, and construct $\mathscr{V}^{\boldsymbol{B}}$ and $\mathscr{L}^{\boldsymbol{B}}$ (for details see [5]).

If $\mathscr{G}$ is $\mathscr{C}$-generic over $\mathfrak{M}$, we define an ultrafilter $\mathscr{F}$ on $\boldsymbol{B}$ as follows:

$$\mathscr{F} = \{b \in \boldsymbol{B}: \exists p \in \mathscr{G}[O_p^{\boldsymbol{B}} \leqslant b]\} \text{ and } p \Vdash \varphi \text{ iff } O_p^{\boldsymbol{B}} \leqslant \|\varphi\|^{\boldsymbol{B}}.$$

Finally, we have $\mathscr{V}^{\boldsymbol{B}}/\mathscr{F} \cong \mathfrak{M}[\mathscr{G}]$.

Proof of Theorem 2. Let f be a function in $\mathfrak{M}[\mathscr{G}]$ with $f\colon [\varkappa]^{<\omega} \to 2$ and let $A = f^{-1}(\{1\})$ be a subset of $[\varkappa]^{<\omega}$ in $\mathfrak{M}[\mathscr{G}]$. There is a function $\check{f}$ in $\mathscr{V}^{\boldsymbol{B}}$ which is the name for f, and $\check{f}\colon [\varkappa]^{<\omega} \to \boldsymbol{B}$, $\check{f}/\mathscr{F} = f$. Let us set $f_n = f|\,[\varkappa]^n$. Since $|\mathscr{C}| = \lambda < \varkappa$, $|\boldsymbol{B}| = 2^\lambda$ and $\varkappa$ is a Ramsey cardinal, $\varkappa$ is strongly inaccessible and $2^\lambda < \varkappa$. Therefore, we can treat $\boldsymbol{B}$ as a subset of $\varkappa$.

Let us consider the structure

$$\mathfrak{X} = \langle \varkappa, \boldsymbol{B}, f_n, \leqslant, \cup, \cap, \ldots, b\rangle_{b \in \boldsymbol{B},\, n \in \omega}.$$

Since $\varkappa \to (\varkappa)^{<\omega}$, we infer from Theorem 2.9 of [2] that there is $X \subset \varkappa$, $|X| = \varkappa$, such that, for any formula $\varphi(v_1, \ldots, v_n)$ and $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$, $\vec{x}, \vec{y} \in [X]^n$ are increasing sequences in the natural ordering of $\varkappa$,

$$\mathfrak{X} \models \varphi(\vec{x}) \text{ iff } \mathfrak{X} \models \varphi(\vec{y})$$

(such a set $\mathfrak{X}$ we call a *set of indiscernibles*).

Now we define the formula

$$\Phi_{b,n}(x_1, \ldots, x_n) = [\tilde{f}_n(x_1, \ldots, x_n) = b]$$

as follows: if $\mathfrak{X} \models \Phi_{b,n}(\vec{x})$ for some $\vec{x} \in [X]^n$, then $\mathfrak{X} \models \Phi_{b,n}(\vec{y})$ for every $\vec{y} \in [X]^n$.

Hence $\check{f} * [X]^n = b$ and $f * [X]^n = 1$. Note that

$$f(x) = \check{f}/\mathscr{F}(x) = \begin{cases} 0 & \text{if } \check{f}(x) \notin \mathscr{F}, \\ 1 & \text{if } \check{f}(x) \in \mathscr{F}, \end{cases}$$

and, for $\vec{x} \in [X]^n$, we have

$$\|(\check{x}_1, \ldots, \check{x}_n) \in A\| = \check{f}(x_1, \ldots, x_n) = b.$$

So

$$\|[X]^n \subset A \vee [X]^n \subset [\varkappa]^n - A\|$$
$$= \prod_{\vec{x} \in [X]^n} \|\vec{x} \in A\| + \prod_{\vec{x} \in [X]^n} \|\vec{x} \notin A\| = b + (-b) = 1$$

which completes the proof.

$\varkappa$ is a *Rowbottom cardinal* if

$$\forall f\colon [\varkappa]^{<\omega} \to \lambda,\ \lambda < \varkappa \Rightarrow \exists X \subset \varkappa \wedge |X| = \varkappa \wedge |f{*}[X]|^{<\omega} = \omega.$$

A cardinal number $\varkappa$ is a Rowbottom cardinal if and only if every two-cardinal structure of type $(\varkappa, \omega)$ has an elementary substructure of type $(\varkappa, \omega)$ (see [9]).

Proof of Theorem 3. Let $f\colon [\varkappa]^{<\omega} \to \lambda$ and $f \in \mathfrak{M}[\mathcal{G}]$ $(f \in \mathcal{V}^B/\mathcal{F})$. Every function from $[\varkappa]^{<\omega}$ to λ in $\mathfrak{M}[\mathcal{G}]$ can be described in $\mathcal{V}^B$ as follows:

Let $f_n = f | [\varkappa]^n$. There is a function $\tilde{f}_n$ in $\mathcal{V}^B$ which is a name for f_n such that

$$\|f_n \text{ is a function mapping } [\varkappa]^n \text{ into } \lambda\| = \mathbf{1}.$$

Let $a_\xi^{\vec{x}} = \|\tilde{f}_n(\vec{x}) = \xi\|$ for $\xi < \lambda$. Then

$$a_\xi^{\vec{x}} \cap a_{\xi'}^{\vec{x}} = 0 \text{ for } \xi \neq \xi' \quad \text{and} \quad \sum_{\xi<\lambda} a_\xi^{\vec{x}} = \mathbf{1}.$$

Since $\boldsymbol{B}$ satisfies ccc, $\{a_\xi^{\vec{x}}\colon \xi < \lambda\}$ is countable and can be indexed by natural numbers (of course, $\sum_{k\in\omega} a_k = \mathbf{1}$).

Let us define a formula Φ_n as follows:

$$\Phi_n(\vec{x}, \xi, a) \text{ iff } \|f_n(\vec{x}) = \xi\| = a.$$

Note that, for each $\vec{x} \in [\varkappa]^n$, we have a countable sequence of pairs $\langle \xi_k^{\vec{x}}, a_k^{\vec{x}} \rangle_{k=0,1}$ such that

$$\|\tilde{f}_n(\vec{x}) = \xi_k^{\vec{x}}\| = a_k^{\vec{x}} \neq 0, \quad \text{where } \xi_k^{\vec{x}} < \lambda.$$

Since $|[\varkappa]^n| = \varkappa$, we have

$$|\xi_k^{\vec{x}}\colon \vec{x} \in [\varkappa]^n, k \in \omega| \leqslant \varkappa \quad \text{and} \quad |a_k^{\vec{x}}\colon \vec{x} \in [\varkappa]^n, k \in \omega| \leqslant \varkappa,$$

and we can treat $\xi_k^{\vec{x}}$ and $a_k^{\vec{x}}$ as elements of $\varkappa$.

Now we define

$$g_1^n(\vec{x}, k) = a_k^{\vec{x}},$$
$$g_2^n(\vec{x}, k) = \xi_k^{\vec{x}}, \quad \text{where } \xi_k^{\vec{x}} < \lambda.$$

Clearly, $\varphi_n\big(\vec{x}, g_2^n(\vec{x}, k), g_1^n(\vec{x}, k)\big)$ holds.

Consider the structure

$$\mathfrak{X} = \langle \varkappa, \lambda, \varphi_n, g_1^n, g_2^n, 0, 1, 2, \ldots \rangle_{n\in\omega}.$$

Since $\varkappa$ is a Rowbottom cardinal, there is an elementary substructure $\mathfrak{A}$ of $\mathfrak{X}$,

$$\mathfrak{A} = \langle A, \lambda \cap A, \varphi_n, g_1^n, g_2^n, 0, 1, 2, \ldots \rangle_{n\in\omega},$$

such that $|A| = \varkappa$, $|\lambda \cap A| = \omega$ (see [9]). Because $\mathfrak{A}$ is an elementary substructure of $\mathfrak{X}$, we have, for each $\vec{x} \in [A]^n$,

$$\mathfrak{X} \models \varphi_n(\vec{x}, g_2^n(\vec{x}, k), g_1^n(\vec{x}, k)) \text{ iff } \mathfrak{A} \models \varphi_n(\vec{x}, g_2^n(\vec{x}, k), g_1^n(\vec{x}, k)),$$

which means that, for each $\vec{x} \in [A]^n$, $\|\tilde{f}_n(\vec{x}) = g_2^n(\vec{x}, k)\| = g_1(\vec{x}, k)$. Hence $\|\tilde{f}_n(\vec{x}) = \xi_k^{\vec{x}}\| = a_k^{\vec{x}}$, $\xi_k^{\vec{x}} < \lambda \cap A$. Since

$$\sum_{k \in \omega} a_k^{\vec{x}} = 1,$$

we have, for each $\vec{x} \in [A]^n$,

$$\|\tilde{f}_n(\vec{x}) \in \check{\lambda} \cap \check{A}\| = \sum_{\xi \in \lambda \cap A} \|\tilde{f}_n(\vec{x}) = \check{\xi}\| \geqslant \sum_{k \in \omega} a_k^{\vec{x}} = 1.$$

Thus

$$\|\tilde{f}_n * [A]^n \subseteq \check{\lambda} \cap \check{A}\| = \prod_{x \in [A]^n} \|\tilde{f}_n(\vec{x}) \in \check{\lambda} \cap \check{A}\| = 1$$

and

$$\|\tilde{f} * [A]^{<\omega} \subseteq \check{\lambda} \cap \check{A}\| = \sum_{n \in \omega} \|\tilde{f}_n * [A]^n \subseteq \check{\lambda} \cap \check{A}\| = 1.$$

Since $\mathscr{V}^B/\mathscr{F} \cong \mathfrak{M}[\mathscr{G}]$, we have $\mathfrak{M}[\mathscr{G}] \models f * [A]^{<\omega} \subseteq \lambda \cap A$. In $\mathfrak{M}$, $|A| = \varkappa$, and $\boldsymbol{B}$ satisfies ccc, so also, in $\mathfrak{M}[\mathscr{G}]$, $|A| = \varkappa$ and $|\lambda \cap A| = \omega$.

COROLLARY. Con(ZFC + *there is a regular Rowbottom cardinal*) $\Rightarrow$ $\Rightarrow$ Con(ZFC + 2^ω *is a Rowbottom cardinal*).

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