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A GENERALIZATION OF THE LERCH–MORDELL FORMULAS FOR POSITIVE DISCRIMINANTS

BY

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1. Introduction. Let K_2, O_F, $\left(\frac{d}{\cdot}\right)$ and $B_{k,\chi}$ denote the Milnor functor, the ring of integers of a number field F, the Kronecker symbol and the k-th Bernoulli number belonging to the character χ, respectively.

For the discriminant d of a real quadratic field and $\chi = \left(\frac{d}{\cdot}\right)$, we put $k_2(d) = B_{2,\chi}$ if $d \neq 5, 8$ and $k_2(5) = k_2(8) = 4$. The Birch–Tate conjecture (the Mazur–Wiles theorem up to 2-torsion, see [4] and [2]) for real quadratic fields F with discriminant d states that $|K_2 O_F| = k_2(d)$.

For the discriminant d of an imaginary quadratic field, let $h(d)$ denote the class number of this field. It is well known that for $d < -4$ and $\chi = \left(\frac{d}{\cdot}\right)$ we have $h(d) = -B_{1,\chi}$. The Lerch–Mordell class number formulas for binary quadratic forms state that, for two relatively prime discriminants $D > 0$, $-\Delta < 0$ of quadratic fields,

$$h(-\Delta D) = 2\sum_{m=1}^{\Delta/2}\left(\frac{-\Delta}{m}\right)\sum_{n=1}^{mD/\Delta}\left(\frac{D}{n}\right),$$

$$-h(-\Delta D) = 2\sum_{m=1}^{D/2}\left(\frac{D}{m}\right)\sum_{n=1}^{m\Delta/D}\left(\frac{-\Delta}{n}\right).$$

Here we put $\sum_{n=1}^{x} = \sum_{n=1}^{[x]}$ ($[x]$ denotes the integer part of x). See [3] and [5].

In the present paper, we generalize these formulas for $k_2(D_1 D_2)$ and $k_2(\Delta_1 \Delta_2)$, where $D_1, D_2 > 0$ and $-\Delta_1, -\Delta_2 < 0$ are pairs of relatively prime discriminants of quadratic fields. We prove the following

THEOREM. *Let $D_1, D_2 > 0$ and $-\Delta_1, -\Delta_2 < 0$ be pairs of relatively prime discriminants of quadratic fields. Then*

$$k_2(D_1D_2) = 4D_1D_2 \sum_{m=1}^{D_2/2} \left(\frac{D_2}{m}\right) \sum_{n=1}^{mD_1/D_2} \left(\frac{D_1}{n}\right)\left(\frac{n}{D_1}-\frac{m}{D_2}\right),$$

$$-k_2(\Delta_1\Delta_2) = 4\Delta_1\Delta_2 \sum_{m=1}^{\Delta_2/2} \left(\frac{-\Delta_2}{m}\right) \sum_{n=1}^{m\Delta_1/\Delta_2} \left(\frac{-\Delta_1}{n}\right)\left(\frac{n}{\Delta_1}-\frac{m}{\Delta_2}\right)$$

$$+2\Delta_1\Delta_2\left(3-\left(\frac{-\Delta_2}{2}\right)\right)h(-\Delta_1)h(-\Delta_2).$$

2. Proof of the Theorem. Let for the discriminant d of a quadratic field and $t \geqslant 0$

$$S_t(d) = |d|^{-t} \sum_{n=1}^{|d|} \left(\frac{d}{n}\right) n^t.$$

Note that for $d < 0$

$$h(d) = -S_1(d), \tag{1}$$

and for $d > 0$

$$k_2(d) = dS_2(d). \tag{2}$$

First, we shall extend results of [5] by proving the following

LEMMA. *Let D_1 and D_2 be relatively prime discriminants of quadratic fields and let $D = D_1D_2$. Put $\Delta = |D|$, $\Delta_i = |D_i|$, $\varepsilon_i = 0$, resp. $\varepsilon_i = 1$, if $D_i > 0$, resp. $D_i < 0$ $(i = 1, 2)$. For an integer $u \geqslant 0$ set*

$$L_u(D) = \sum_{m=1}^{\Delta_2} \left(\frac{D_2}{m}\right) \sum_{n=1}^{m\Delta_1/\Delta_2} \left(\frac{D_1}{n}\right)\left(\frac{n}{\Delta_1}-\frac{m}{\Delta_2}\right)^u.$$

Then

$$(-1)^{\varrho} S_t(D) = (-1)^{t+\varepsilon_2} \sum_{s=0}^{t-1} (-1)^s \binom{t}{s} \sum_{u=0}^{s} (-1)^u \binom{s}{u} L_u(D)$$

$$+\sum_{u=0}^{t} \binom{t}{u} S_u(D_1) S_{t-u}(D_2),$$

where $\varrho = 1$ if $D_1, D_2 < 0$, and $\varrho = 0$ otherwise.

Proof. Let $D_1, \ldots, D_r$ denote discriminants relatively prime in pairs and let μ of them be negative. Put $D = D_1 \ldots D_r$, $\Delta_i = |D_i|$ for $i = 1, \ldots, r$ and $\Delta = |D|$. Mordell [5] has noticed that the complete set of non-negative residues $\bmod \Delta$ is given by

$$R = \Delta\left(\frac{n_1}{\Delta_1}+\ldots+\frac{n_r}{\Delta_r}\right) \quad \text{with } 0 \leqslant n_i < \Delta_i,\ i = 1, \ldots, r.$$

Then putting $n = R - E\Delta$, where $E = [R/\Delta]$, we have (see [5])

$$\left(\frac{D}{n}\right) = (-1)^{[\mu/2]}\left(\frac{D_1}{n_1}\right)\cdots\left(\frac{D_r}{n_r}\right),$$

and

$$S_t := S_t(D) = (-1)^{[\mu/2]}\Delta^{-t}\sum_{\substack{0<n_i<\Delta_i\\ i=1,\dots,r}}\left(\frac{D_r}{n_1}\right)\cdots\left(\frac{D_1}{n_r}\right)(R-E\Delta)^t$$

$$= (-1)^{[\mu/2]}\Delta^{-t}\sum_{\substack{0<n_i<\Delta_i\\ i=1,\dots,r}}\left(\frac{D_1}{n_1}\right)\cdots\left(\frac{D_r}{n_r}\right)\sum_{s=0}^{t}\binom{t}{s}R^s(-1)^{t-s}E^{t-s}\Delta^{t-s}$$

$$= (-1)^{[\mu/2]+t}\sum_{s=0}^{t}(-1)^s\binom{t}{s}\sum_{\substack{0<n_i<\Delta_i\\ i=1,\dots,r}}\left(\frac{D_1}{n_1}\right)\cdots\left(\frac{D_r}{n_r}\right)\left(\frac{R}{\Delta}\right)^s E^{t-s}.$$

Hence for $r = 2$ we obtain

$$(-1)^{\varrho+t+\varepsilon_2}S_t = \sum_{s=0}^{t-1}(-1)^s\binom{t}{s}\sum_{\substack{0<n_i<\Delta_i, i=1,2\\ n_1/\Delta_1<n_2/\Delta_2}}\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{n_2}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}+1\right)^s$$

$$+(-1)^{t+\varepsilon_2}\sum_{\substack{0<n_i<\Delta_i\\ i=1,2}}\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{n_2}\right)\left(\frac{n_1}{\Delta_1}+\frac{n_2}{\Delta_2}\right)^t$$

$$= \sum_{s=0}^{t-1}(-1)^s\binom{t}{s}\sum_{u=0}^{s}(-1)^u\binom{s}{u}$$

$$\times\sum_{n_2=1}^{\Delta_2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{D_1}{n_1}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}\right)^u$$

$$+(-1)^{t+\varepsilon_2}\sum_{u=0}^{t}\binom{t}{u}\sum_{\substack{0<n_i<\Delta_i\\ i=1,2}}\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{n_2}\right)\left(\frac{n_1}{\Delta_1}\right)^u\left(\frac{n_2}{\Delta_2}\right)^{t-u}$$

$$= \sum_{s=0}^{t-1}(-1)^s\binom{t}{s}\sum_{u=0}^{s}(-1)^u\binom{s}{u}L_u(D)$$

$$+(-1)^{\varrho}\sum_{u=0}^{t}\binom{t}{u}S_u(D_1)S_{t-u}(D_2).$$

Thus the Lemma is proved.

To obtain the Theorem we apply the Lemma to formula (2).

First let $D = D_1 D_2$ and $D_1, D_2 > 0$. Then $\varrho = 0$. By $S_j(D_i) = 0$ for $i = 1, 2$ and $j = 0, 1$, we get

$$S_2 = -L_0(D)+2L_1(D)$$
$$= -\sum_{n_2=1}^{D_2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2D_1/D_2}\left(\frac{D_1}{n_1}\right)+2\sum_{n_2=1}^{D_2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2D_1/D_2}\left(\frac{D_1}{n_1}\right)\left(\frac{n_1}{D_1}-\frac{n_2}{D_2}\right).$$

For a non-integer a, put $\sum\limits_{n=a}^{x} = \sum\limits_{n=[a]+1}^{x}$. Since for any discriminant d and a non-integer a, $1 \leqslant a \leqslant |d|$,

$$\sum_{n=a}^{|d|}\left(\frac{d}{n}\right) = \sum_{n=1}^{a}\left(\frac{d}{n}\right), \tag{3}$$

we obtain

$$\sum_{n_2=1}^{D_2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2D_1/D_2}\left(\frac{D_1}{n_1}\right) = 0.$$

Therefore we have

$$\frac{1}{2}S_2 = \sum_{n_2=1}^{D_2/2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2D_1/D_2}\left(\frac{D_1}{n_1}\right)\left(\frac{n_1}{D_1}-\frac{n_2}{D_2}\right)$$
$$+\sum_{n_2=1}^{D_2/2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=n_2D_1/D_2}^{D_1-1}\left(\frac{D_1}{n_1}\right)\left(\frac{D_1-n_1}{D_1}-\frac{D_2-n_2}{D_2}\right)$$
$$= 2\sum_{n_2=1}^{D_2/2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{n_2D_1/D_2}\left(\frac{D_1}{n_1}\right)\left(\frac{n_1}{D_1}-\frac{n_2}{D_2}\right).$$

If $D = \Delta_1\Delta_2$ and $-\Delta_1, -\Delta_2 < 0$, then $\varrho = 1$. In view of (1) in this case we have

$$-S_2 = -L_0(D)+2L_1(D)+2h(-\Delta_1)h(-\Delta_2)$$
$$= -\sum_{n_2=1}^{\Delta_2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)$$
$$+2\sum_{n_2=1}^{\Delta_2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}\right)+2h(-\Delta_1)h(-\Delta_2).$$

Therefore, by the equality

$$\sum_{n_2=1}^{\Delta_2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)=0$$

(this is a consequence of (3)), we get

$$-\frac{1}{2}S_2=\sum_{n_2=1}^{\Delta_2/2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}\right)$$

$$+\sum_{n_2=1}^{\Delta_2/2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=n_2\Delta_1/\Delta_2}^{\Delta_1-1}\left(\frac{-\Delta_1}{n_1}\right)\left(\frac{\Delta_1-n_1}{\Delta_1}-\frac{\Delta_2-n_2}{\Delta_2}\right)$$

$$+h(-\Delta_1)h(-\Delta_2)$$

$$=\sum_{n_2=1}^{\Delta_2/2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}\right)$$

$$+\sum_{n_2=1}^{\Delta_2/2}\left(\frac{-\Delta_2}{n_2}\right)\left[-\frac{n_2}{\Delta_2}\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)+h(-\Delta_1)\right.$$

$$\left.+\frac{1}{\Delta_1}\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)n_1\right]+h(-\Delta_1)h(-\Delta_2)$$

$$=2\sum_{n_2=1}^{\Delta_2/2}\left(\frac{-\Delta_2}{n_2}\right)\sum_{n_1=1}^{n_2\Delta_1/\Delta_2}\left(\frac{-\Delta_1}{n_1}\right)\left(\frac{n_1}{\Delta_1}-\frac{n_2}{\Delta_2}\right)$$

$$+2h(-\Delta_1)h(-\Delta_2)\left(3-\left(\frac{-\Delta_2}{2}\right)\right)$$

because of (1) and of the following well-known formula which is true for any $d<0$:

$$h(d)=\left(2-\left(\frac{d}{2}\right)\right)^{-1}\sum_{l=1}^{|d|/2}\left(\frac{d}{l}\right).$$

Thus the Theorem is proved.

3. Remarks. In the paper we have used an elementary idea of [5]. We can prove the Theorem using formulas (12.1) and (12.2) in [1]. There exist analogous formulas for $D=D_1\dots D_r$ with $r>2$ which can be obtained by using the same methods.

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