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$\aleph_0$-*CATEGORICITY OF PRODUCTS OF* 1-*UNARY ALGEBRAS*

BY

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Recent papers by Waszkiewicz and Węglorz [8], Waszkiewicz [7], and the author [3] have been concerned with product operations which preserve $\aleph_0$-categoricity. At the other extreme is the free product operation $*$, say for groups: if $\mathfrak{A}$ and $\mathfrak{B}$ are both not the 1-element group, then $\mathfrak{A}*\mathfrak{B}$ is not $\aleph_0$-categorical. However, the infinite weak direct power or direct multiple operation (denoted by $\oplus_\omega\mathfrak{A}$) is at neither extreme. There is the very elegant result of Rosenstein [4]: let $\mathfrak{A}$ be a finite group; then $\oplus_\omega\mathfrak{A}$ is $\aleph_0$-categorical iff $\mathfrak{A}$ is Abelian. It is the purpose of this paper* to investigate the infinite direct multiple operation on 1-unary algebras. It is shown that the $\aleph_0$-categoricity of $\mathfrak{A}$ is neither necessary nor sufficient for the $\aleph_0$-categoricity of $\oplus_\omega\mathfrak{A}$. A sufficient condition on $\mathfrak{A}$ is then given, and some necessary conditions are also indicated. There are some preliminary results on the $\aleph_0$-categoricity of the product of two 1-unary algebras. In Jónsson [2] there can be found an introduction to 1-unary algebras (e.g., p. 27 and 69-71) and also several results concerning them.

A 1-*unary algebra* $\mathfrak{A} = \langle A, f\rangle$ is a non-empty set A with a one-place function f from A into A. A 1-*unary algebra with zero*, denoted by $\mathfrak{A} = \langle A, f, 0\rangle$, is a 1-unary algebra with a distinguished element 0 such that $f(0) = 0$. In what follows all algebras are finite or denumerably infinite. The cardinality of a set S is denoted by $|S|$.

Definitions. (1) The *core* of $\mathfrak{A}$, denoted by $C(\mathfrak{A})$, is

$$\{x \in A \mid (\exists n > 0)(f^n(x) = x)\}.$$

(2) $\mathfrak{A}$ is *limited* if there exists an N such that $n \leqslant N$ whenever $x, f(x), f^2(x), \ldots, f^n(x)$ are all different.

(3) $K(x) = \{y \mid (\exists n \geqslant 0)(f^n(y) = x)\}$.

(4) $K(x)$ has *finite depth* t if

$$(\exists y \in K(x))(f^t(y) = x) \quad \text{and} \quad (\forall z)(\forall n > t)(f^n(z) \neq x).$$

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(5) $\mathfrak{A}$ is *connected* if $(\forall x)(\forall y)(\exists m)(\exists n)(f^m(x) = f^n(y))$.

(6) A *branch* in $\mathfrak{A}$ of length n ending at x is a sequence $y, f(y), \ldots, f^n(y)$ of different elements of $\mathfrak{A}$ such that $n \geqslant 0$, $f^n(y) = x$ and $f^{-1}(y) = \emptyset$.

(7) $B(\mathfrak{A})$ is the *length* of the longest branch in $\mathfrak{A}$ ending at some $x \in C(\mathfrak{A})$ and intersecting $C(\mathfrak{A})$ only at x. ($B(\mathfrak{A})$ does not always exist.)

(8) If $\mathfrak{A} = \langle A, f, 0\rangle$ is a 1-unary algebra with 0, we define its *denumerable direct multiple* $\oplus_\omega \mathfrak{A} = \langle \oplus_\omega A, f, 0\rangle$ as the subalgebra of the direct power $\mathfrak{A}^\omega$ consisting of those $a\colon \omega \to A$ such that $\{i \mid a(i) \neq 0\}$ is finite. We have $0(i) = 0$ for all $i < \omega$. Note that $\oplus_\omega A$ is a countable set.

A countable algebra $\mathfrak{A}$ is $\aleph_0$-*categorical* if it is determined up to isomorphism ($\cong$) among the countable algebras by its theory in the elementary first-order predicate calculus. A result which will be used frequently is the following theorem of Shishmarev [6]. Its proof uses a result of Ryll-Nardzewski [5].

THEOREM (Shishmarev). *Let $\mathfrak{A}$ be a countable 1-unary algebra. Then $\mathfrak{A}$ is $\aleph_0$-categorical iff $\mathfrak{A}$ is limited and, to within isomorphism, only a finite number of $K(x)$ are realized in $\mathfrak{A}$.*

There are several simple facts about 1-unary algebras which will be often used. They are collected, without a proof, in the following

LEMMA. *Let $\mathfrak{A} = \langle A, f\rangle$ be a 1-unary algebra.*

(1) *$\mathfrak{A}$ is the disjoint union of its maximal connected subalgebras.*

(2) *$B(\mathfrak{A}) = 0$ iff $\mathfrak{A} = C(\mathfrak{A})$ iff f is one-to-one.*

(3) *Assume that $\mathfrak{A}$ is limited. Then*

(i) *$B(\mathfrak{A})$ is finite and $C(\mathfrak{A}) \neq \emptyset$;*

(ii) *$K(x)$ has finite depth iff $x \notin C(\mathfrak{A})$;*

(iii) *$x \in C(\mathfrak{A})$ iff $K(x)$ is a maximal connected subalgebra;*

(iv) *there exists an N such that if $\mathfrak{A}'$ is a maximal connected subalgebra, then $0 < |C(\mathfrak{A}')| \leqslant N$.*

Let $\mathfrak{A} = \langle A, f, 0\rangle$ be a 1-unary algebra with 0.

(4) *If $\mathfrak{A}'$ is the maximal connected subalgebra containing 0, then $C(\mathfrak{A}') = \{0\}$.*

(5) *$a \in C(\oplus_\omega \mathfrak{A})$ iff $a(i) \in C(\mathfrak{A})$ for all $i < \omega$.*

(6) *If $a, \beta \in \oplus_\omega \mathfrak{A}$, $a(k) = \beta(j)$, $a(j) = \beta(k)$, and $a(i) = \beta(i)$ for $i \notin \{j, k\}$, then $K(a) \cong K(\beta)$.*

(7) *If $\mathfrak{A}$ is limited, then $B(\mathfrak{A}) = B(\oplus_\omega \mathfrak{A})$.*

It follows from [1] or [8] that if $\mathfrak{A}$ and $\mathfrak{B}$ are $\aleph_0$-categorical, then the product $\mathfrak{A} \times \mathfrak{B}$ is $\aleph_0$-categorical. However, the converse is false.

Example. We define $\mathfrak{A} = \langle A, f\rangle$ which is not $\aleph_0$-categorical, but $\mathfrak{A} \times \mathfrak{A}$ is $\aleph_0$-categorical. Let

$$A = \{a_n\}_{n<\omega} \cup \{b_n^m\}_{n<\omega, m<\omega} \cup \{c_n^m\}_{n<\omega, m<\omega} \cup \{d_n^i\}_{n<\omega, i<n} \cup \{{}_m e_n^i)_{n<\omega, m<\omega, i<n}.$$

For all n, m, $i < \omega$ write

$$f({}_m e_n^i) = d_n^i, \quad f(d_n^i) = f(a_n) = a_n, \quad f(c_n^m) = b_n^m, \quad f(b_n^m) = a_n.$$

Clearly, if $j \neq k$, then $K(a_j) \cong K(a_k)$ is not true. So $\mathfrak{A}$ is not $\aleph_0$-categorical by the theorem of Shishmarev. Just as $\mathfrak{A}$ is the disjoint union of the maximal connected subalgebras $K(a_j)$, $\mathfrak{A} \times \mathfrak{A}$ is the disjoint union of the maximal connected subalgebras $K(\langle a_j, a_k\rangle)$. But for any j and k it can be seen that the subalgebra $K(\langle a_j, a_k\rangle)$ has the core $\{\langle a_j, a_k\rangle\}$ and, outside the core, consists of an infinite set of γ such that $f(\gamma) = \langle a_j, a_k\rangle$, $f^{-1}(\gamma)$ is infinite and $f^{-2}(\gamma) = \emptyset$, with an infinite set of δ such that $f(\delta) = \langle a_j, a_k\rangle$, $f^{-1}(\delta)$ is a 1-element set, and $f^{-2}(\delta) = \emptyset$. It follows by Shishmarev's theorem that $\mathfrak{A} \times \mathfrak{A}$ is $\aleph_0$-categorical.

Definition. $\mathfrak{A} = \langle A, f\rangle$ has no infinite branching if $f^{-1}(x)$ is a finite set whenever $x \in A$.

THEOREM 1. *Suppose $\mathfrak{A} = \langle A, f\rangle$ and $\mathfrak{B} = \langle B, f\rangle$ have no infinite branching. Then $\mathfrak{A} \times \mathfrak{B}$ is $\aleph_0$-categorical iff both $\mathfrak{A}$ and $\mathfrak{B}$ are $\aleph_0$-categorical.*

Proof. From right to left the implication follows from [1] or [8].

Assume $\mathfrak{A} \times \mathfrak{B}$ is $\aleph_0$-categorical. So $\mathfrak{A} \times \mathfrak{B}$ is limited. But if $a, f(a), \ldots, f^n(a)$ are different members of A, then, for any $b \in B$, $\langle a, b\rangle, f(\langle a, b\rangle), \ldots, f^n(\langle a, b\rangle)$ are different members of $A \times B$. Hence $\mathfrak{A}$ (and similarly $\mathfrak{B}$) is limited. We claim that there exist an M such that $|f^{-1}(x)| \leqslant M$ for all $x \in A$. If it is false, then we can choose a sequence $\{a_i\}_{i<\omega} \subseteq A$ such that $|f^{-1}(a_i)| \geqslant i$. Let $b \in B$ satisfy $f^{-1}(b) \neq \emptyset$. In $\mathfrak{A} \times \mathfrak{B}$ we then have

$$i \leqslant |f^{-1}(\langle a_i, b\rangle)| < \aleph_0.$$

So $\{K(\langle a_i, b\rangle)\}$ has an infinite subset of pairwise non-isomorphic members, contradicting the $\aleph_0$-categoricity of $\mathfrak{A} \times \mathfrak{B}$. By lemma (3) (iv), we know that there exists an N such that the core of each maximal connected subalgebra has not greater than N members. Also, by lemma (3) (i), $B(\mathfrak{A})$ is finite. From these results it follows that the number of different $K(x)$ which could be realized in $\mathfrak{A}$ is finite. So $\mathfrak{A}$ (and similarly $\mathfrak{B}$) is $\aleph_0$-categorical.

We now turn to the question of a necessary and sufficient condition on $\{\mathfrak{A}, \mathfrak{B}\}$ that $\mathfrak{A} \times \mathfrak{B}$ be $\aleph_0$-categorical. There is such a condition, but it is very tedious to state. The condition requires that each of $\mathfrak{A}$, $\mathfrak{B}$ be limited (say by N) and each of a sequence $D_1, D_2, \ldots, D_N$ of conditions be satisfied. As an illustration we state D_2.

D_2. There exist positive integers M_1, M_2 and $\{N_i\}_{i \leqslant M_2}$ with the following property:

Let $a \in A$, $b \in B$ and let

$$n(m) = \left|\{x \mid f(x) = a \text{ and } |f^{-1}(x)| = m\}\right|$$

and

$$r(t) = \big|\{x \mid f(x) = b \text{ and } |f^{-1}(x)| = t\}\big|,$$

where $m, t, n(m), r(t) \leqslant \aleph_0$. Then

(1) $n(0)\cdot r(0) + \sum\{n(0)\cdot r(t) \mid t \leqslant \aleph_0\} + \sum\{r(0)\cdot n(m) \mid m \leqslant \aleph_0\}$ is either not greater than M_1 or equal to $\aleph_0$;

(2) if $n(m) > 0$ and $r(t) > 0$, then $m\cdot t$ is either not greater than M_2 or equal to $\aleph_0$; and

(3) $\sum\{n(m)\cdot r(t) \mid m\cdot t = i\}$ is either not greater than N_i or equal to $\aleph_0$.

We could then prove the following

LEMMA. *$\mathfrak{A} \times \mathfrak{B}$ has a finite number of non-isomorphic $K(x)$ of depth 2 iff D_2 is true of $\{\mathfrak{A}, \mathfrak{B}\}$.*

We now consider the question of the $\aleph_0$-categoricity of infinite direct multiples $\oplus_\omega \mathfrak{A}$ of a 1-unary algebra $\mathfrak{A}$ with 0. We first show that the $\aleph_0$-categoricity of $\mathfrak{A}$ neither implies nor is implied by the $\aleph_0$-categoricity of $\oplus_\omega \mathfrak{A}$.

Example. Put $A = \{a, b, 0\}$ and $f(0) = 0$, $f(a) = f(b) = a$, and let $\mathfrak{A} = \langle A, f, 0\rangle$. Since A is finite, $\mathfrak{A}$ is $\aleph_0$-categorical. But $\oplus_\omega \mathfrak{A}$ is not $\aleph_0$-categorical because if $a_n \in \oplus_\omega A$ is given by $a_n(i) = a$ for $i < n$ and by $a_n(i) = 0$ for $i \geqslant n$, then $|f^{-1}(a_n)| = 2^n$. So $\{K(a_n)\}_{n<\omega}$ are pairwise non-isomorphic.

Example. Put $A = \{a_n\}_{n<\omega} \cup \{b_n^i\}_{n<\omega, i<n} \cup \{0, c_0, c_1\}$, and, for all n, i, $f(b_n^i) = f(a_n) = a_n$, $f(c_i) = f(0) = 0$, and let $\mathfrak{A} = \langle A, f, 0\rangle$. Since $|f^{-1}(a_n)| = n+1$, $\mathfrak{A}$ is not $\aleph_0$-categorical. But if $a \in \oplus_\omega A$, then either $f^{-1}(a) = \emptyset$ or $f^{-1}(a)$ is infinite and $f^{-2}(a) = f^{-1}(a)$. So, to within isomorphism, there are only two $K(x)$ realized in $\oplus_\omega \mathfrak{A}$, and thus $\oplus_\omega \mathfrak{A}$ is $\aleph_0$-categorical.

It is not hard to show that if f is one-to-one, then $\mathfrak{A}$ is $\aleph_0$-categorical iff $\oplus_\omega \mathfrak{A}$ is.

The following theorem provides a sufficient condition on $\mathfrak{A}$ that $\oplus_\omega \mathfrak{A}$ be $\aleph_0$-categorical.

THEOREM 2. *Let $\mathfrak{A} = \langle A, f, 0\rangle$ be a 1-unary algebra with 0, and let $\mathfrak{A}'$ denote its maximal connected subalgebra containing 0. If $\mathfrak{A}$ is limited and there exists $c \in \mathfrak{A}'$, $c \neq 0$, $f(c) = 0$, such that every branch in $\mathfrak{A}'$ ending at 0 and passing through c has length $B(\mathfrak{A})$, then $\oplus_\omega \mathfrak{A}$ is $\aleph_0$-categorical.*

Assume $\mathfrak{A}$ satisfy the conditions above. We first prove the following

LEMMA. *Suppose $\beta \in \oplus_\omega A$ and $K(\beta)$ has finite depth t. Then*

(1) if $f(\gamma) = \beta$, then there are infinitely many different γ_i such that $f(\gamma_i) = \beta$ and $K(\gamma_i) \cong K(\gamma)$; and

(2) *if* $b \in A$ *and every branch in* $\mathfrak{A}$ *ending at* b *has length not less than* t *and if* $\beta' \in \oplus_\omega A$ *and* $i_0 < \omega$ *are such that* $\beta'(i_0) = b$, $\beta(i_0) = 0$ *and, for all* $i \neq i_0$, $\beta(i) = \beta'(i)$, *then* $K(\beta) \cong K(\beta')$.

Proof. We establish (1) and (2) by simultaneous induction on t. If $t = 0$, then $f^{-1}(\beta) = \emptyset$, $K(\beta) = \{\beta\}$, and so $K(\beta') = \{\beta'\}$. Hence (2) is true. Also, (1) is trivially true since no such γ could exist. We now assume that $t \geqslant 1$ and that (1) and (2) are true for all non-negative integers less than t.

First we prove (1). Since $K(\beta)$ has finite depth, $\beta \notin C(\oplus_\omega \mathfrak{A})$ and $t < B(\oplus_\omega \mathfrak{A}) = B(\mathfrak{A})$. Moreover, $K(\gamma)$ has depth not greater than $t-1$, and, by hypothesis, any branch in $\mathfrak{A}$ ending at c has length $B(\mathfrak{A})-1 > t-1$. Let J be an infinite subset of ω such that if $j \in J$, then $\gamma(j) = 0$. Let γ_j be defined by $\gamma_j(i) = \gamma(i)$ if $i \neq j$, and $\gamma_j(j) = c$. By (2) of the induction hypothesis, we get $K(\gamma) \cong K(\gamma_j)$. Since $f(c) = 0$, we have $f(\gamma_j) = \beta$. Clearly, if $j \neq k$, then $\gamma_j \neq \gamma_k$. So we have (1).

Now we prove (2). Note that $K(\beta')$ also has depth t. From (1), as proved for t, we infer that if $f(\gamma') = \beta'$, then there are infinitely many different γ'_j such that $f(\gamma'_j) = \beta'$ and $K(\gamma'_j) \cong K(\gamma')$. Of course, the same holds for β. Hence, to obtain $K(\beta) \cong K(\beta')$, it remains to show that for any $K(x)$

$$(\exists\gamma)\big(f(\gamma) = \beta \text{ and } K(\gamma) \cong K(x)\big) \text{ iff } (\exists\gamma')\big(f(\gamma') = \beta' \text{ and } K(\gamma') \cong K(x)\big).$$

Suppose $f(\gamma) = \beta$. So $K(\gamma)$ has depth not greater than $t-1$. By lemma (6), we can assume $\gamma(i_0) = 0$. Let $d \in A$ satisfy $f(d) = b$. We know that such a d exists since every branch in $\mathfrak{A}$ ending at b has length not less than $t \geqslant 1$. Furthermore, every branch in $\mathfrak{A}$ ending at d has length not less than $t-1$. Define γ' by $\gamma'(i_0) = d$ and by $\gamma'(i) = \gamma(i)$ for $i \neq i_0$. Then $f(\gamma') = \beta'$ and, by the induction hypothesis on (2), $K(\gamma) \cong K(\gamma')$.

Now suppose $f(\gamma') = \beta'$. Thus $f\big(\gamma'(i_0)\big) = \beta'(i_0) = b$. Define γ by $\gamma(i_0) = 0$ and by $\gamma(i) = \gamma'(i)$ for $i \neq i_0$. Then $f(\gamma) = \beta$, and $K(\gamma)$ has depth not greater than $t-1$. Also, every branch in $\mathfrak{A}$ ending at $\gamma'(i_0)$ has length not less than $t-1$. Hence, by the induction hypothesis on (2), $K(\gamma) \cong K(\gamma')$.

Corollary. *Suppose* $a \in C(\oplus_\omega \mathfrak{A})$, $\beta, \beta' \in \oplus_\omega \mathfrak{A} - C(\oplus_\omega \mathfrak{A})$, $f(\beta) = f(\beta') = a$, $a(i_0) = \beta(i_0) = 0$, $\beta'(i_0) = c$ *and, for* $i \neq i_0$, $\beta(i) = \beta'(i)$. *Then* $K(\beta) \cong K(\beta')$.

Proof. Since $\beta \notin C(\oplus_\omega \mathfrak{A})$, $K(\beta)$ has finite depth t, and

$$t \leqslant B(\oplus_\omega \mathfrak{A}) - 1 = B(\mathfrak{A}) - 1.$$

Since every branch in $\mathfrak{A}$ ending at c has length $B(\mathfrak{A})-1$, lemma (2) gives the result.

Proof of theorem 2. Since $\mathfrak{A}$ is limited, so is $\oplus_\omega \mathfrak{A}$, and hence, by lemmas (3) (iv) and (5), there is a bound on the size of the core of any maximal connected subalgebra of $\oplus_\omega \mathfrak{A}$. Thus $B(\oplus_\omega \mathfrak{A})$ is finite. By the above-given lemma and its corollary, we infer that if $f(\beta) = a$ and $\beta \notin C(\oplus_\omega \mathfrak{A})$, then

$$\{\gamma \mid f(\gamma) = a \text{ and } K(\gamma) \cong K(\beta)\}$$

is infinite. A straightforward arithmetical computation now shows that if p_i denotes the number of non-isomorphic $K(x)$ of finite depth i which could be realized in $\oplus_\omega \mathfrak{A}$, then $p_0 = 1$, $p_1 = 1$ and, for $i \geqslant 2$,

$$p_i = 2^{p_0} \cdot 2^{p_1} \cdot \ldots \cdot 2^{p_{i-2}} \cdot (2^{p_{i-1}} - 1).$$

For $a \in C(\oplus_\omega \mathfrak{A})$, let

$$K'(a) = \{a\} \cup \big\{\beta \mid (\exists n \geqslant 1) \ \big(f^n(\beta) = a \text{ and } \beta, f(\beta), f^2(\beta), \ldots, f^{n-1}(\beta) \notin C(\oplus_\omega \mathfrak{A})\big)\big\}.$$

Informally, $K'(a)$ is the tree, rooted at a, and intersecting the core of $\oplus_\omega \mathfrak{A}$ only at a. Then the number of non-isomorphic $K'(a)$ that are realized in $\oplus_\omega \mathfrak{A}$ is at most $\sum_{j=0}^{M} p_j$, where $M = B(\oplus_\omega \mathfrak{A})$. So, to within isomorphism, there are only a finite number of maximal connected subalgebras in $\oplus_\omega \mathfrak{A}$. Hence $\oplus_\omega \mathfrak{A}$ is $\aleph_0$-categorical.

It is easy to show that if $\oplus_\omega \mathfrak{A}$ is $\aleph_0$-categorical, then $\mathfrak{A}$ is limited and there exists a branch ending at 0 of length $B(\mathfrak{A})$. In fact, much more than this could be shown to be necessary. Furthermore, the sufficient condition of the theorem is not necessary because of the example that follows below. We have been unable to close the gap between the sufficient condition of the theorem and the above necessary conditions, even in the case where A is a finite set.

The $\oplus_\omega \mathfrak{A}$, which are shown above to be $\aleph_0$-categorical, have the property that if $f(x) = y$ and $x \notin C(\oplus_\omega \mathfrak{A})$, then

$$\{z \mid f(z) = y \text{ and } K(z) \cong K(x)\}$$

is infinite. This is not a necessary condition, as shown by the following example:

Example. Put $A = \{0, b, a_1, a_2, a_3, a_4\}$ and $f(a_i) = a_{i-1}$ for $i = 2$, 3, 4, $f(b) = a_1, f(a_1) = f(0) = 0$, and let $\mathfrak{A} = \langle A, f, 0\rangle$. We omit the details but it can be shown that $\oplus_\omega \mathfrak{A}$ is a connected $\aleph_0$-categorical algebra with $B(\oplus_\omega \mathfrak{A}) = 4$. Let a and β be given by $a(0) = a_1$, $\beta(0) = a_2$ and, for $i > 0$, $a(i) = \beta(i) = 0$. Then $f(\beta) = a$, $\beta \notin C(\oplus_\omega \mathfrak{A})$ and β is the unique element in $\oplus_\omega A$ such that $f(\beta) = a, f^{-1}(\beta) \neq \emptyset$ and, for $f(\gamma) = \beta$, we have $f^{-1}(\gamma) \neq \emptyset$. So $\{z \mid f(z) = a \text{ and } K(z) \cong K(\beta)\}$ is a 1-element set.

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