COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 1

A DUEL WITH SILENT-NOISY GUN VERSUS NOISY GUN*

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1. A duel is a game involving two players, each with a gun and one or several bullets, who start a fixed distance apart and walk towards each other, firing his bullet(s) at the other player at such times as to maximize his chance of hitting his opponent and minimize his own chance of being hit. More specifically, if the players are A and B, the payoff of the duel is $+1$ if A survives and B is hit, -1 if B survives and A is hit, and 0 if both or neither survive. A's object is to maximize the expected payoff and B's object is to minimize it.

The accuracy function of each player is the probability of his hitting his opponent by a shot as a function of the time elapsed from the moment the players start walking until the shot is fired. It is assumed the players walk towards each other with constant velocity, and the total time it would take the players to meet is 1. The accuracy functions, $P(t)$ and $Q(t)$ for A and B respectively, are assumed to be everywhere continuous and differentiable with a positive derivative everywhere and such that

$$P(0) = Q(0) = 0, \quad P(1) = Q(1) = 1.$$

In some duels, a player knows when his opponent has fired a bullet by its noise, and in others he does not know because the shot is silent. Among the duels already solved are the noisy-vs-noisy, noisy-vs-silent, and silent-vs-silent duels with one bullet for each contestant, by Blackwell and Girschick [1]. The n bullets -vs- m bullets noisy duel was solved by Blackwell and Girschick [1] for the case of equal accuracy functions. The n bullets -vs- m bullets silent duel was solved by Restrepo [3]. Other references to work on duels can be found in Karlin [2] in Chapters 5 and 6 and the notes at the end of the chapters.

* Prepared with partial support of the National Science Foundation and Office of Naval Research of the United States Government.

The duel analyzed in this paper is a silent-noisy -vs- noisy duel. Player A has a silent bullet and a noisy bullet which must be fired in that order, and player B has one noisy bullet. A strategy for A consists of a joint probability density on $[0,1]\times[0,1]$ for its silent and noisy shots with zero probability for the noisy shot to be fired before or simultaneous with the silent shot, and a strategy for B consists of a single probability density on $[0,1]$ for its noisy shot. The players are assumed to choose their firing spots via their respective probability densities and fire accordingly unless the opponent fires and misses his noisy shot while the first player still has a bullet left. In that case, it is assumed the player waits until $t=1$ for a sure hit. Clearly in any optimal strategies this wait until $t=1$ after the opponent has missed his noisy shot is necessary and will be assumed without further mention in all strategies for the rest of this paper.

The payoff function will be designated by $Z[S_A, S_B]$, where S_A and S_B are strategies for A and B respectively, and it equals

$$\text{Prob}\,\{A \text{ alone survives}\} - \text{Prob}\,\{B \text{ alone survives}\}$$

when S_A and S_B are adopted.

The duel is said to *have a value* λ if

$$\max_{S_A\in\mathscr{A}} \min_{S_B\in\mathscr{B}} Z[S_A, S_B] = \min_{S_B\in\mathscr{B}} \max_{S_A\in\mathscr{A}} Z[S_A, S_B] = \lambda,$$

where $\mathscr{A}$ and $\mathscr{B}$ are the sets of A and B strategies respectively.

If the duel has a value λ, S_{A*} is said to be an *optimal strategy for* A if

$$\min_{S_B\in\mathscr{B}} Z[S_{A*}, S_B] = \lambda.$$

Similarly S_{B*} is said to be an *optimal strategy for* B if

$$\max_{S_A\in\mathscr{A}} Z[S_A, S_{B*}] = \lambda.$$

It follows from the above that if strategies S_{A*} and S_{B*} can be found such that

$$\min_{S_B\in\mathscr{B}} Z[S_{A*}, S_B] = \max_{S_A\in\mathscr{A}} [S_A, S_{B*}],$$

then S_{A*} and S_{B*} are optimal and the value of the duel is $Z[S_{A*}, S_{B*}]$.

$\bar{S}_A$ will be said to be *optimal against* $\bar{S}_B$ if

$$Z[\bar{S}_A, \bar{S}_B] = \max_{S_A\in\mathscr{A}} [S_A, \bar{S}_B]$$

and similarly for $\bar{S}_B$ optimal against $\bar{S}_A$.

It will be convenient to occasionally represent a strategy for B by c, or F, if B shoots at c with probability one, or shoots via the distribution function F, and to represent a strategy for A by (b, c), or (F, c), if A shoots his silent and noisy shots at b and c with probability one, or shoots his silent shot via the distribution function F and shoots his noisy shot at c with probability one. In each case the meaning of the notation will be readily apparent.

In the following sections, two particular strategies for A and B will be shown to be optimal, and the value of the duel will be computed. Then it will be shown that these strategies are the only optimal strategies. The conclusion contains some remarks on related unsolved problems.

2. There is an optimal strategy for each player. The strategy for A consists of an absolutely continuous distribution over an interval for the firing of his silent bullet followed by a fixed point where A always fires his noisy bullet. The strategy for B consists of an absolutely continuous distribution over the same interval as for A's silent bullet plus a discrete part at the same point where A fires his noisy bullet.

Before describing the optimal strategies, two constants x_0 and a will first be defined. Since $P(t)$ and $Q(t)$ both increase continuously and monotonically from 0 to 1 there is a unique solution to the equation

$$P(x)+Q(x) = 1.$$

Let x_0 be this solution. Clearly $0 < x_0 < 1$.

The expression

$$\int_c^{x_0} \frac{Q'(u)Q(c)}{Q^2(u)P(u)}\, du$$

is a monotonically decreasing function of c for $c \in (0, x_0)$. This can be seen from

$$\begin{aligned}\frac{d}{dc}\left[Q(c)\int_c^{x_0} \frac{Q'(u)\,du}{Q^2(u)P(u)}\right] &= Q'(c)\int_c^{x_0} \frac{Q'(u)\,du}{Q^2(u)P(u)} + Q(c)\left(-\frac{Q'(c)}{Q^2(c)P(c)}\right)\\ &= Q'(c)\int_c^{x_0} \frac{Q'(u)\,du}{Q^2(u)P(u)} - \frac{Q'(c)}{Q(c)P(c)}\\ &< Q'(c)\left[\frac{1}{P(c)}\int_c^{x_0} \frac{Q'(u)\,du}{Q^2(u)} - \frac{1}{Q(c)P(c)}\right]\\ &= -\frac{Q'(c)}{P(c)Q(x_0)} < 0.\end{aligned}$$

Also

$$\lim_{c\to 0}\int\limits_c^{x_0} \frac{Q'(u)Q(c)}{Q^2(u)P(u)}\,du > 1.$$

This is seen from

$$\int\limits_c^{x_0} \frac{Q'(u)Q(c)}{Q^2(u)P(u)}\,du \geqslant \frac{Q(c)}{P(x_0)}\int\limits_c^{x_0}\frac{Q'(u)}{Q^2(u)}\,du$$

$$= \frac{1}{P(x_0)}\left[1-\frac{Q(c)}{Q(x_0)}\right] \to \frac{1}{P(x_0)} > 1 \quad \text{as } c\to 0.$$

Since this expression equals 0 for $c = x_0$, there is a unique number a for which

(1) $$\int\limits_a^{x_0} \frac{Q'(u)Q(a)}{Q^2(u)P(u)}\,du = 1, \quad 0 < a < x_0.$$

The strategies D_A for A and D_B for B are now defined as follows:

D_A consists of A firing his silent bullet in the interval $[a, x_0)$ with probability density

$$f(x) = \frac{Q'(x)Q(a)}{Q^2(x)P(x)}$$

and firing his noisy bullet at x_0. Equation (1) insures that this is a well-defined strategy.

D_B consists of B firing his bullet in $[a, x_0)$ with probability density

$$g(x) = \beta\,\frac{(1-v_0)}{2}\,\frac{P'(x)P(x_0)}{P^2(x)Q(x)}$$

where $v_0 = P(x_0)-Q(x_0) < 1$ and

$$\beta = \frac{1}{1+\dfrac{1-v_0}{2}P(x_0)\displaystyle\int\limits_a^{x}\frac{P'(u)\,du}{P^2(u)Q(u)}}$$

and firing his bullet at x_0 with probability β. Note that from the definition of x_0

(2) $$v_0 = P(x_0)-Q(x_0) = 1-2Q(x_0) = 2P(x_0)-1.$$

That this is a well-defined strategy is seen from

$$\beta+\int\limits_a^{x_0} g(u)\,du = \beta\left[1+\frac{(1-v_0)}{2}P(x_0)\int\limits_a^{x_0}\frac{P'(u)\,du}{P^2(u)Q(u)}\right] = 1.$$

D_B is optimal against D_A. This is seen from

Case 1) $c < a$

$$Z[D_A, c] = \text{Prob }\{B \text{ misses at } c\} - \text{Prob }\{B \text{ hits at } c\}$$
$$= 1 - Q(c) - Q(c) = 1 - 2Q(c) > 1 - 2Q(a).$$

Case 2) $c \in [a, x_0)$

$$\begin{aligned} Z[D_A, c] &= \text{Prob }\{A \text{ hits before } c \text{ fires}\} + \\ &\quad + [\text{Prob }\{A \text{ does not hit before } c \text{ fires}\}] \times \\ &\quad \times [\text{Prob }\{B \text{ misses at } c\} - \text{Prob }\{B \text{ hits at } c\}] \\ &= \int_a^c f(u)P(u)\,du + \left[1 - \int_a^c f(u)P(u)\,du\right][-Q(c) + 1 - Q(c)] \\ &= Q(a)\int_a^c \frac{Q'(u)\,du}{Q^2(u)} + [1 - 2Q(c)]\left[1 - Q(a)\int_a^c \frac{Q'(u)\,du}{Q^2(u)}\right] \\ &= 1 - 2Q(a) \end{aligned}$$

Case 3) First notice that if both duelists have a single bullet and both fire at x_0 the payoff is

$$P(x_0)[1 - Q(x_0)] - [1 - P(x_0)]Q(x_0) = P(x_0) - Q(x_0) = v_0.$$

Thus

$$\begin{aligned} Z[D_A, x_0] &= \int_a^{x_0} f(u)P(u)\,du + \left[1 - \int_a^{x_0} f(u)P(u)\,du\right]v_0 \\ &= (1 - v_0)\int_a^{x_0} \frac{Q(a)Q'(u)}{Q^2(u)}\,du + v_0 = 1 - 2Q(a). \end{aligned}$$

Case 4) $c > x_0$

$$Z[D_A, c] = \int_a^{x_0} f(u)P(u)\,du + \left[1 - \int_a^{x_0} f(u)P(u)\,du\right],$$
$$\left[P(x_0) - \left(1 - P(x_0)\right)\right] = 1 - 2Q(a)$$

since $2P(x_0) - 1 = v_0$, resulting in the same expression as in case 3. Thus

$$\min_c Z[D_A, c] = 1 - 2Q(a)$$

and if B shoots his bullet with distribution function $F(t)$,

$$Z[D_A, F(t)] = \int_0^1 Z[D_A, u]\,dF(u) \geqslant 1 - 2Q(a).$$

Therefore the best B can do is $1-2Q(a)$, and since D_B is, by cases 2 and 3, a combination of strategies, all of which give payoff $1-2Q(a)$, D_B is optimal against D_A.

D_A is optimal against D_B, for suppose A fires his silent bullet at b and his noisy bullet at $c > b$.

Case 1) $c < x_0$.

If $c < a$, A clearly improves his payoff by setting $c = a$. So suppose $c \geqslant a$. Similarly, if $b < a$, A improves by setting $b = a$. So $a \leqslant b \leqslant c < x_0$ may be assumed. (Equality of b and c is allowed in this case solely for convenience in the above reasoning.)

$$\begin{aligned} Z[(b,c),D_B]&-Z[(b,x_0),D_B]\\ &=\Big\{-\int_a^b g(u)Q(u)\,du+\int_a^b g(u)[1-Q(u)]\,du+P(b)\Big[1-\int_a^b g(u)\,du\Big]+\\ &\quad+[1-P(b)]\Big[-\int_b^c g(u)Q(u)\,du+\int_b^c g(u)[1-Q(u)]\,du+\\ &\quad+\big(P(c)-1+P(c)\big)\Big(\int_c^{x_0} g(u)\,du+\beta\Big)\Big]\Big\}-\\ &\quad-\Big\{-\int_a^b g(u)Q(u)\,du+\int_a^b g(u)[1-Q(u)]\,du+P(b)\Big[1-\int_a^b g(u)\,du\Big]+\\ &\quad+[1-P(b)]\Big[-\int_b^{x_0} g(u)Q(u)\,du+\int_b^{x_0} g(u)\big(1-Q(u)\big)\,du+\beta v_0\Big]\Big\}\\ &=[1-P(b)]\Big[\{P(c)-[1-P(c)]\}\Big\{\int_c^{x_0} g(u)\,du+\beta\Big\}-\\ &\quad-\Big\{-\int_c^{x_0} g(u)Q(u)\,du+\int_c^{x_0} g(u)[1-Q(u)]\,du+\beta v_0\Big\}\Big]. \end{aligned}$$

The expression in the brackets at the right equals

$$\begin{aligned} &\big(2P(c)-1\big)\Big(\int_c^{x_0} g(u)\,du+\beta\Big)-\Big(\int_c^{x_0} g(u)\,du-2\int_c^{x_0} g(u)Q(u)\,du+\beta v_0\Big)\\ &\quad=\Big(\int_c^{x_0} g(u)\,du\Big)\big(2P(c)-2\big)+2\int_c^{x_0} g(u)Q(u)\,du+\beta\big(2P(c)-2P(x_0)\big)\\ &\quad<2\Big(P(c)\int_c^{x_0} g(u)\,du-\int_c^{x_0} g(u)[1-Q(u)]\,du\Big)\\ &\quad<2\Big\{[1-Q(x_0)]\int_c^{x_0} g(u)\,du-\int_c^{x_0} g(u)[1-Q(u)]\,du\Big\}<0. \end{aligned}$$

Since $1-P(b)>0$, this shows $Z[(b,c),D_B]<Z[(b,x_0),D_B]$.

Case 2) $c = x_0$.

If $b < a$, clearly A's payoff can be increased by setting $b = a$. So it may be assumed $b \in [a, x_0)$.

A preliminary equation will first be derived,

$$\text{(3)} \qquad \int_a^{x_0} \frac{P'(u)du}{P^2(u)Q(u)} = \left[-\frac{1}{P(u)Q(u)}\right]\Big|_a^{x_0} - \int_a^{x_0} \frac{Q'(u)du}{P(u)Q^2(u)}$$

$$= \frac{1}{Q(a)P(a)} - \frac{1}{Q(x_0)P(x_0)} - \frac{1}{Q(a)}$$

from equation (1) and integration by parts.

$$Z[(b, x_0), D_B] = -\int_a^b g(u)Q(u)du + \int_a^b g(u)[1-Q(u)]du +$$

$$+\left[1-\int_a^b g(u)du\right]P(b) - [1-P(b)]\int_b^{x_0} g(u)Q(u)du +$$

$$+[1-P(b)]\int_b^{x_0} g(u)\big(1-Q(u)\big)du + [1-P(b)]\beta v_0$$

$$= -\int_a^{x_0} g(u)Q(u)du + \int_a^{x_0} g(u)du + \beta v_0 - \int_a^{x_0} g(u)Q(u)du +$$

$$+P(b)\left[1-\int_a^{x_0} g(u)du + 2\int_b^{x_0} g(u)Q(u)du - \beta v_0\right].$$

This last expression in brackets equals

$$\beta - \beta v_0 + 2\beta\left(\frac{1-v_0}{2}\right)\left(\frac{P(x_0)}{P(a)}\right)P(a)\left(-\frac{1}{P(x_0)} + \frac{1}{P(b)}\right) = \frac{\beta(1-v_0)P(x_0)}{P(b)}$$

and so

$$Z[(b, x_0), D_B]$$

$$= -\beta\left(\frac{1-v_0}{2}\right)P(x_0)\int_a^{x_0} 2(Q)(u)\frac{P'(u)du}{P^2(u)Q(u)} + 1 - \beta + \beta v_0 + \beta(1-v_0)P(x_0)$$

$$= 1 + \beta(1-v_0)\left(P(x_0) - \frac{P(x_0)}{P(a)}\right).$$

By the definition of β and equation (3) this equals

$$1+\frac{(1-v_0)\left(P(x_0)-\frac{P(x_0)}{P(a)}\right)}{1+\frac{1-v_0}{2}P(x_0)\left(\frac{1}{Q(a)P(a)}-\frac{1}{Q(x_0)(Px_0)}-\frac{1}{Q(a)}\right)}$$

$$=1+\frac{\left(2-2P(x_0)\right)\left(P(x_0)-\frac{P(x_0)}{P(a)}\right)}{1+\left(1-P(x_0)\right)P(x_0)\left(\frac{1}{Q(a)P(a)}-\frac{1}{[1-P(x_0)]P(x_0)}-\frac{1}{Q(a)}\right)}$$

$$=1+\frac{2P(x_0)\left(1-P(x_0)\right)\left(1-\frac{1}{P(a)}\right)}{\left(1-P(x_0)\right)P(x_0)\frac{1}{Q(a)}\left(\frac{1}{P(a)}-1\right)}=1-2Q(a).$$

Case 3) $c>x_0$, $b<x_0$.
As in cases 1 and 2 it may be supposed that $b\in[a,x_0)$.

$$Z[(b,c),D_B]=-\int_a^b g(u)Q(u)\,du+\int_a^b g(u)\left(1-Q(u)\right)du+$$

$$+P(b)\left[1-\int_a^b g(u)\,du\right]+[1-P(b)]\left[-\int_b^c g(u)Q(u)\,du+\right.$$

$$\left.+\int_b^c g(u)\left(1-Q(u)\right)du+\beta\left(-Q(x_0)+1-Q(x_0)\right)\right].$$

But since $-Q(x_0)+1-Q(x_0)=v_0$, this expression is the same as the one in case 2, and thus equals $1-2Q(a)$.
Case 4) $b=x_0$, $c>x_0$.

$$Z[(b,c),D_B]=-\int_a^{x_0} g(u)Q(u)\,du+\int_a^{x_0} g(u)\left(1-Q(u)\right)du+$$

$$+\beta\left[\left(1-Q(x_0)\right)-Q(x_0)\left(1-P(x_0)\right)\right]$$

$$=\int_a^{x_0} g(u)-2\int_a^{x_0} g(u)Q(u)\,du+\beta\left(v_0+Q(x_0)P(x_0)\right)$$

$$= 1-\beta-2\beta\left(\frac{1-v_0}{2}\right)P(x_0)\int_a^{x_0}\frac{P'(u)\,du}{P^2(u)}+\beta\big(v_0+Q(x_0)P(x_0)\big)$$

$$= 1-\beta-\beta(1-v_0)P(x_0)\left(-\frac{1}{P(x_0)}+\frac{1}{P(a)}\right)+\beta\big(v_0+Q(x_0)P(x_0)\big)$$

$$= 1-\beta(1-v_0)\frac{P(x_0)}{P(a)}+\beta P(x_0)Q(x_0)$$

$$< 1-\beta(1-v_0)\frac{P(x_0)}{P(a)}+2\beta P(x_0)Q(x_0)$$

$$= 1-\beta(1-v_0)P(x_0)\left(\frac{1}{P(a)}-1\right)$$

$$= 1-\frac{(1-v_0)P(x_0)\left(\dfrac{1}{P(a)}-1\right)}{1+\dfrac{1-v_0}{2}P(x_0)\left(\dfrac{1}{Q(a)P(a)}-\dfrac{1}{Q(x_0)P(x_0)}-\dfrac{1}{Q(a)}\right)}$$

$$= 1-\frac{(1-v_0)P(x_0)\left(\dfrac{1}{P(a)}-1\right)}{\dfrac{(1-v_0)P(x_0)}{2Q(a)}\left(\dfrac{1}{P(a)}-1\right)} = 1-2Q(a)$$

from the definition of β and equation (3).

Case 5) $b, c > x_0$.

$$Z[(b,c), D_B] = -\int_a^{x_0} g(u)Q(u)\,du+\int_a^{x_0} g(u)\big(1-Q(u)\big)\,du+\beta\big(1-Q(x_0)-Q(x_0)\big)$$

which is smaller than the first expression in case 4, and thus smaller than $1-2Q(a)$.

Thus the best A can do against D_B is to choose probabilistically from strategies of cases 2 and 3. Since D_A is a combination of strategies of case 2, D_A is optimal against D_B.

D_A and D_B are optimal against each other and so by the remarks in the introduction they are optimal strategies and the value of the game is

$$Z[D_A, D_B] = 1-2Q(a).$$

3. It will now be shown that D_A is the only optimal strategy for A. By the above analysis of cases 1 through 5 it is apparent that in any optimal strategy for A, A must shoot his silent bullet in $[a, x_0)$ with probability one and must shoot his noisy bullet in $[x_0, 1]$ with probability one.

It shall first be shown that in an optimal strategy for A he must shoot his noisy bullet at x_0 with probability one. For suppose A always shoots his silent bullet in $[a, x_0)$ with probability one, and if $F(x)$ is the probability A shoots his noisy bullet before x suppose $F(x_0) = 0$ and $F(x_0+) < 1$. Call this strategy S_A.

Define S_{B_ε}, $0 < \varepsilon < 1-x_0$, to be the strategy for B consisting of the density function $g(x)$ over $[a, x_0)$ for the probability of B's shooting there and the probability β for B's shooting at $x_0+\varepsilon$.

The conditional payoff

$$Z[S_A, S_{B_\varepsilon}, \text{given } B \text{ fires in } [a, x_0)]$$

is defined to be the probabilistic average of Z when S_A and S_{B_ε} are played, averaged only over results where B fires in $[a, x_0)$, and scaled up by the reciprocal of the probability that B fires in $[a, x_0)$ as in the computation of conditional probabilities. Other conditional payoffs are computed similarly.

$$\begin{aligned}
Z[S_A, S_{B_\varepsilon}]-Z[S_A, D_B]&\\
&= (1-\beta)\big(Z[S_A, S_{B_\varepsilon}, \text{given } B \text{ fires in } [a, x_0)]\big)\\
&\quad +\beta(Z[S_A, S_{B_\varepsilon}, \text{given } B \text{ fires at } x_0+\varepsilon])-\\
&\quad -(1-\beta)\big(Z[S_A, D_B, \text{given } B \text{ fires in } [a, x_0)\big)+\\
&\quad +\beta(Z[S_A, D_B, \text{given } B \text{ fires at } x_0])\\
&= \beta(Z[S_A, S_{B_\varepsilon}, \text{given } B \text{ fires at } x_0+\varepsilon]-\\
&\quad -Z[S_A, D_B, \text{given } B \text{ fires at } x_0])\\
&= \beta(Z[S_A, S_{B_\varepsilon}, \text{given } A \text{ misses silent and } B \text{ fires at } x_0+\varepsilon]-\\
&\quad -Z[S_A, D_B, \text{given } A \text{ misses silent and } B \text{ fires at } x_0]).
\end{aligned}$$

Letting ε always be chosen so that the probability A fires his noisy shot at $x_0+\varepsilon$ is zero, the expression in brackets equals

$$\Big\{\int_{x_0}^{x_0+\varepsilon} P(u)\,dF(u)-\int_{x_0}^{x_0+\varepsilon}\big(1-P(u)\big)\,dF(u)+\big(1-2Q(x_0+\varepsilon)\big)\int_{x_0+\varepsilon}^{1} dF(u)\Big\}-$$

$$-\Big\{-Q(x_0)\big(1-P(x_0)\big)F(x_0+)+\big(1-Q(x_0)\big)P(x_0)F(x_0+)+\big(1-2Q(x_0)\big)\int_{x_0+}^{1} dF(u)\Big\}$$

$$= -\int_{x_0}^{x_0+\varepsilon} dF(u)+2\int_{x_0}^{x_0+\varepsilon} P(u)\,dF(u)+\int_{x_0+\varepsilon}^{1} dF(u)-2Q(x_0+\varepsilon)\int_{x_0+\varepsilon}^{1} dF(u)-$$

$$-\bigl(2P(x_0)-1\bigr)F(x_0+)-\int_{x_0+}^{1}dF(u)+2Q(x_0)\int_{x_0+}^{1}dF(u)$$

$$=2\int_{x_0+}^{x_0+\varepsilon}\bigl(P(u)-P(x_0)\bigr)dF(u)+2\bigl(Q(x_0)-Q(x_0+\varepsilon)\bigr)\int_{x_0+\varepsilon}^{1}dF(u)$$

$$\sim Q\Bigl(2\varepsilon P'(x_0)\int_{x_0+}^{x_0+\varepsilon}dF(u)-2\varepsilon Q'(x_0)\int_{x_0+\varepsilon}^{1}dF(u)\Bigr)$$

as $\varepsilon\to 0$, using equation (2) to obtain the next to last expression.

If ε is chosen small enough so that

$$\frac{\int_{x_0+\varepsilon}^{1}dF(u)}{\int_{x_0+}^{x_0+\varepsilon}dF(u)}>\frac{P'(x_0)}{Q'(x_0)},$$

then the last expression in parentheses is negative.

Thus it has been shown that an optimal strategy for A will always have the silent bullet fired in $[a, x_0)$ and the noisy bullet at x_0.

Suppose in an optimal strategy S_A for A, the probability that A fires his silent shot at a is $p>0$. Then by the following

LEMMA. *Let S_A and S_B be optimal strategies and a be in the support of S_B. Then $Z[S_A, a]$ equals the value of the game. If (b, c) is in the support of S_A, then $Z[(b, c), S_B]$ equals the value of the game.*

whose proof can be found in Karlin [2], p. 27, and the fact that a is in the support of D_B,

$$Z[S_A, a] = -Q(a)\bigl[1-p+p\bigl(1-P(a)\bigr)\bigr]+1-Q(a)$$
$$= 1-2Q(a)+p\bigl(1-P(a)-Q(a)\bigr)>1-2Q(a).$$

Thus in an optimal strategy A cannot shoot his silent shot at a with positive probability.

To see that there is a unique optimal probability distribution on $[a, x_0)$ for A's silent bullet, let $F(t)$ be the distribution function of the silent bullet on $[a, x_0)$ for an optimal A strategy S_F. Since D_B has each point of $[a, x_0)$ for its support, if $a\leqslant x_1<x_2<x_0$ where x_1 and x_2 are chosen so that the probability that A fires at x_1 or x_2 is zero, then

$$Z[S_F, x_1] = Z[S_F, x_2].$$

Expanding each side of this equation,

$$\int_a^{x_1} P(u)\,dF(u) - Q(x_1)\Big(1-\int_a^{x_1} P(u)\,dF(u)\Big) + \big(1-Q(x_1)\big)\Big(1-\int_a^{x_1} P(u)\,dF(u)\Big)$$

$$= \int_a^{x_2} P(u)\,dF(u) - Q(x_2)\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big) + \big(1-Q(x_2)\big)\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big);$$

$$\int_{x_1}^{x_2} P(u)\,dF(u) = \big(2Q(x_2)-1\big)\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big) - \big(2Q(x_1)-1\big)\times$$

$$\times\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big) + \big(2Q(x_1)-1\big)\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big) -$$

$$-\big(2Q(x_1)-1\big)\Big(1-\int_a^{x_1} P(u)\,dF(u)\Big)$$

$$= 2\big(Q(x_2)-Q(x_1)\big)\Big(1-\int_a^{x_2} P(u)\,dF(u)\Big) + \big(2Q(x_1)-1\big)\times$$

$$\times\Big(-\int_{x_1}^{x_2} P(u)\,dF(u)\Big);$$

$$Q(x_1)\int_{x_1}^{x_2} P(u)\,dF(u) = [Q(x_2)-Q(x_1)]\Big[1-\int_a^{x_2} P(u)\,dF(u)\Big]. \tag{4}$$

If x is any point in (a, x_0), by letting x_1 and x_2 tend to x from below and above in equation (4) we see the probability that A fires at x is zero. Hence there are no restrictions on x_1 or x_2 in equation (4) other than $a \leqslant x_1 < x_2 < x_0$. Dividing by $x_2 - x_1$ and letting x_1 and x_2 tend to x from below and above,

$$Q(x)P(x) \lim_{x_1\uparrow x,\, x_2\downarrow x} \frac{\int_{x_1}^{x_2} dF(u)}{x_2-x_1} = Q'(x)\Big[1-\int_a^{x} P(u)\,dF(u)\Big].$$

This shows that $\int_a^x dF(u)$ is a differentiable function of x for x in $[a, x_0)$, and

$$f^*(x) \equiv F'(x) = \frac{Q'(x)}{Q(x)P(x)}\Big[1-\int_a^{x} P(u)\,dF(u)\Big].$$

Hence

$$P(x)f^*(x) = \frac{Q'(x)}{Q(x)}\Big[1-\int_a^x P(u)f^*(u)\,du\Big].$$

Setting

$$h(x) = \int_a^x P(u)f^*(u)\,du, \qquad h'(x)+\frac{Q'(x)}{Q(x)}h(x) = \frac{Q'(x)}{Q(x)}.$$

The solution for which $h(a) = 0$ is

$$h(x) = 1-\frac{Q(a)}{Q(x)}, \qquad f^*(x) = \frac{h'(x)}{P(x)} = \frac{Q(a)Q'(x)}{Q^2(x)P(x)}$$

from the definition of $h(x)$, and hence $f^*(x) = f(x)$ and D_A is the unique optimal strategy for A.

It will now be shown that D_B is the unique optimal strategy for B.

By case 1 under the proof that D_B is optimal against D_A it is clear that in any optimal strategy for B, the probability that B fires before a is zero.

Also in any optimal strategy for B the probability that B fires at a is 0. For suppose S_B is an optimal B strategy with the probability that B fires at a is $c > 0$. Since all of $[a, x_0)$ is in the support of D_A's silent density,

$$\begin{aligned}
0 &= Z[(a, x_0), S_B] - \lim_{x \downarrow a} Z[(x, x_0), S_B] \\
&= \big\{-cQ(a)\big(1-P(a)\big)+cP(a)\big(1-Q(a)\big)+ \\
&\quad +(1-c)Z[(a, x_0), S_B, \text{given } B \text{ does not fire at } a]\big\} - \\
&\quad -\big\{-cQ(a)+c\big(1-Q(a)\big)+(1-c)Z[(a, x_0), S_B, \text{given } B \text{ does not fire at } a]\big\} \\
&= cP(a)Q(a)-c\big(1-Q(a)\big)\big(1-P(a)\big) > cP(a)Q(a)-cP(a)Q(a) = 0.
\end{aligned}$$

An analysis exactly analogous to that showing A must shoot his noisy shot at x_0 in an optimal strategy will also show B must never shoot his noisy shot after x_0 in an optimal strategy.

To obtain the distribution function for B's shot let S_B be an optimal B strategy and let $x_1 \uparrow x$ and $x_2 \downarrow x$ through values for which the probability B fires at x_1 or x_2 is zero in S_B. Since all of $[a, x_0)$ is in the support of A's silent density,

$$Z[(x_1, x_0), S_B] = Z[(x_2, x_0), S_B].$$

Letting $G(x)$ be the probability B fires before x, this becomes

$$-\int_a^{x_1} Q(u)\,dG(u)+\int_0^{x_1}\big(1-Q(u)\big)\,dG(u)+P(x_1)\int_{x_1}^{x_0+} dG(u)+$$
$$+\big(1-P(x_1)\big)\Big(-\int_{x_1}^{x_0} Q(u)\,dG(u)+\int_{x_1}^{x_0}\big(1-Q(u)\big)\,dG(u)\Big)+$$
$$+\big(1-P(x_1)\big)v_0\big(G(x_0+)-G(x_0)\big)$$
$$=-\int_0^{x_2} Q(u)\,dG(u)+\int_0^{x_2}\big(1-Q(u)\big)\,dG(u)+P(x_2)\int_{x_2}^{x_0+} dG(u)+$$
$$+\big(1-P(x_2)\big)\Big(-\int_{x_2}^{x_0} Q(u)\,dG(u)+\int_{x_2}^{x_0}\big(1-Q(u)\big)\,dG(u)\Big)+$$
$$+\big(1-P(x_2)\big)v_0\big(G(x_0+)-G(x_0)\big).$$

Collecting terms on the left side, this becomes

$$\big(P(x_2)-P(x_1)\big)v_0\big(G(x_0+)-G(x_0)\big)+2P(x_1)\int_{x_1}^{x_2} Q(u)\,dG(u)+$$
$$+\big(P(x_1)-P(x_2)\big)\Big(\int_{x_2}^{x_0+} dG(u)-\int_{x_2}^{x_0}\big(1-2Q(u)\big)\,dG(u)\Big)=0.$$

Dividing by x_2-x_1 and taking the limit,

$$P'(x)v_0\big(G(x_0+)-G(x_0)\big)-P'(x)\Big(\int_x^{x_0+} dG(u)-\int_x^{x_0}\big(1-2Q(u)\big)\,dG(u)\Big)$$
$$=-2P(x)Q(x)\lim_{x_1\uparrow x,\,x_2\downarrow x}\frac{\int_{x_1}^{x_2} dG(u)}{x_2-x_1}.$$

This shows that in S_B the probability B fires at x is zero for all x in $[a, x_0)$ and that $\int_a^x dG(u)$ is a differentiable function of x for $x\in(a, x_0)$. Its derivative is

$$g^*(x)\equiv G'(x)=\frac{P'(x)}{2P(x)Q(x)}\Big\{(1-v_0)\big(G(x_0+)-G(x_0)\big)+2\int_x^{x_0} Q(u)\,dG(u)\Big\}.$$

Setting

$$h(x)=\int_a^x Q(u)g(u)\equiv\int_a^x Q(u)\,dG(u),\qquad x\in[a, x_0),$$

we have

$$h' = \frac{P'(x)}{2P(x)}\{(1-v_0)(G(x_0+)-G(x_0))+2(h(x_0)-h(x))\},$$

$$h'(x)+\frac{P'(x)}{P(x)}h(x) = \frac{P'(x)}{P(x)}\left\{\frac{1-v_0}{2}\beta^*+h(x_0)\right\},$$

where

$$\beta^* = G(x_0+)-G(x_0).$$

The solution of this equation for which $h(a) = 0$ is

$$(5)\qquad \begin{aligned} h(x) &= \left\{\frac{1-v_0}{2}\beta^*+h(x_0)\right\}\left\{1-\frac{P(a)}{P(x)}\right\},\\ g^*(x) &= \frac{h'(x)}{Q(x)} = \left\{\frac{1-v_0}{2}\beta^*+h(x_0)\right\}\frac{P(a)P'(x)}{P^2(x)Q(x)}. \end{aligned}$$

Clearly

$$(6)\qquad \int_a^{x_0} g^*(u)\,du+\beta^* = \left[\frac{1-v_0}{2}\beta^*+h(x_0)\right]P(a)\int_a^{x_0}\frac{P'(u)\,du}{P^2(u)Q(u)}+\beta^* = 1$$

and from equation (5)

$$(7)\qquad h(x_0) = \left[\frac{1-v_0}{2}\beta^*+h(x_0)\right]\left[1-\frac{P(a)}{P(x_0)}\right].$$

Solving equations (6) and (7) for β and $h(x_0)$

$$\beta^* = \frac{1}{1+\dfrac{1-v_0}{2}P(x_0)\displaystyle\int_a^{x_0}\frac{P'(u)\,du}{P^2(u)Q(u)}} = \beta,$$

$$h(x_0) = \frac{1-v_0}{2}\beta^*\left[\frac{P(x_0)}{P(a)}-1\right].$$

Therefore

$$\begin{aligned} g^*(x) &= \left\{\frac{1-v_0}{2}\beta^*+\frac{1-v_0}{2}\beta^*\left[\frac{P(x_0)}{P(a)}-1\right]\right\}\frac{P(a)P'(x)}{P^2(x)Q(x)}\\ &= \frac{1-v_0}{2}\beta\frac{P(x_0)P'(x)}{P^2(x)Q(x)} = g(x) \end{aligned}$$

and S_B is D_B, the unique optimal strategy for B.

4. The noisy-silent-vs-noisy duel where the first player must fire his noisy shot first has as its solution an easy extension of the silent -vs- noisy duel, solved by Blackwell and Girschick [1]. The density function for the first player's silent shot is the same in the noisy-silent-vs-noisy and the silent-vs-noisy duels, and the first player always shoots his noisy shot at a point a fixed distance ahead of where his silent density begins. The second player shoots at this advanced point with a certain probability, and if he doesn't shoot there his conditional density for the rest of the duel is the same as his density in the silent-vs-noisy duel.

An interesting unsolved duel is the silent-noisy-vs-noisy where the first player may shoot either shot first such that when the second player hears the first player's noisy shot he doesn't know whether or not the first player still has his silent shot left. (**P 586**)

There also remains the still unsolved m by n noisy-vs-noisy duel, with not necessarily equal accuracy functions. In fact, it is not known whether this duel always has a value. If it did then a recursive process could be used to solve each duel, by using the solution of the m by $n-1$ and $m-1$ by n duels to solve the m by n duel. (**P 587**)

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Reçu par la Rédaction le 30. 10. 1965