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WEAKLY SIGN-SYMMETRIC MATRICES AND SOME DETERMINANTAL INEQUALITIES*

BY

DAVID CARLSON (CORVALLIS, OREGON, U. S. A.)

1. We shall discuss some related sets of inequalities which are common to the theories of totally non-negative matrices (cf. [1], vol. II, p. 98), M-matrices (cf. [5]), and positive semi-definite matrices (either real symmetric or complex hermitian). First, we need some definitions: if $A = [a_{ij}]$, $i, j = 1, \ldots, m$, with complex elements, then let

$$A\begin{bmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{bmatrix} = [a_{i_p j_q}], \quad p, q = 1, \ldots, k,$$

$$A\begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{pmatrix} = \det A\begin{bmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{bmatrix}.$$

Also, we will let $\{\alpha\,\beta\,\gamma\}$ denote any partition of a subset of $\{1 \ldots n\}$ into three disjoint ordered subsets (not necessarily non-empty). If, for example, $\alpha = (i_1 \ldots i_p)$, $\beta = (j_1 \ldots j_q)$, $\gamma = (k_1 \ldots k_r)$, we will for brevity write $\alpha\beta\gamma$ for the ordered subset $(i_1 \ldots i_p\; j_1 \ldots j_q\; k_1 \ldots k_r)$. If $\alpha = \emptyset$, we define

$$A\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = 1.$$

The set of inequalities we shall discuss are

$$A\begin{pmatrix} \beta\gamma \\ \beta\gamma \end{pmatrix} \leqslant A\begin{pmatrix} \beta \\ \beta \end{pmatrix} A\begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \quad \text{for all } \{\alpha = \emptyset, \beta, \gamma\} \tag{1}$$

and a generalization of (1),

$$A\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} A\begin{pmatrix} \alpha\,\beta\,\gamma \\ \alpha\,\beta\,\gamma \end{pmatrix} \leqslant A\begin{pmatrix} \alpha\,\beta \\ \alpha\,\beta \end{pmatrix} A\begin{pmatrix} \alpha\,\gamma \\ \alpha\,\gamma \end{pmatrix} \quad \text{for all } \{\alpha, \beta, \gamma\}. \tag{2}$$

A unification of several aspects of the theories of positive matrices and positive definite symmetric matrices was posed as a research problem

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by Taussky [6], who mentioned an unpublished unification by Ky Fan of the proofs of (1) for totally non-negative and positive definite symmetric matrices. A proof of (1) which holds for all nonsingular matrices in the classes mentioned above had previously essentially been given by Gantmacher and Kreĭn ([2], p. 111-117); this note is based on their work.

We define A to be *weakly sign-symmetric* if

$$A\begin{pmatrix}\alpha\,\beta\\ \alpha\,\gamma\end{pmatrix}A\begin{pmatrix}\alpha\,\gamma\\ \alpha\,\beta\end{pmatrix}\geqslant 0 \quad \text{for all } \{\alpha,\beta=(i),\gamma=(j)\}. \tag{3}$$

This is equivalent to the definition given by Kotelyanskiĭ [3]. We note that clearly all totally non-negative and positive semi-definite matrices are weakly sign-symmetric. It is proved by Ostrowski in [5] that, if A is an M-matrix, then every principal submatrix of A is also an M-matrix, and further, that each element of $\operatorname{Adj} A$ is nonnegative. This implies that all M-matrices are weakly sign-symmetric.

2. Gantmacher and Kreĭn prove ([2], p. 116)

$$A\begin{pmatrix}1\dots m\\ 1\dots m\end{pmatrix}\leqslant A\begin{pmatrix}1\dots p\\ 1\dots p\end{pmatrix}A\begin{pmatrix}p+1\dots m\\ p+1\dots m\end{pmatrix},\quad 1\leqslant p<m, \tag{*}$$

under the assumptions that all principal minors of A are positive and

$$A\begin{pmatrix}i_1\dots i_s\\ k_1\dots k_s\end{pmatrix}A\begin{pmatrix}k_1\dots k_s\\ i_1\dots i_s\end{pmatrix}\geqslant 0, \tag{**}$$

$$1\leqslant i_1<\dots<i_s\leqslant m,\quad 1\leqslant k_1<\dots<k_s\leqslant m,$$

whenever $\sum|i_\nu-k_\nu|=1$. As pointed out by Kotelyanskiĭ, (cf. [1], vol II, p. 103) these conditions are not sufficient; for example, if

$$A=\begin{bmatrix}2&0&1\\ 0&2&0\\ -1&0&2\end{bmatrix},\quad 10=A\begin{pmatrix}1\ 2\ 3\\ 1\ 2\ 3\end{pmatrix}>A\begin{pmatrix}1\\ 1\end{pmatrix}A\begin{pmatrix}2\ 3\\ 2\ 3\end{pmatrix}=8.$$

Their proof of (*), however, is valid, if we assume (**) whenever exactly one of $i_1-k_1,\dots,i_s-k_s$ is non-zero. As a corollary ([2], p. 117) of this result, they prove that

$$A\begin{pmatrix}1\dots p\ \ p+1\dots q\ \ q+1\dots m\\ 1\dots p\ \ p+1\dots q\ \ q+1\dots m\end{pmatrix}A\begin{pmatrix}p+1\dots q\\ p+1\dots q\end{pmatrix} \tag{***}$$

$$\leqslant A\begin{pmatrix}1\dots q\\ 1\dots q\end{pmatrix}A\begin{pmatrix}p+1\dots n\\ p+1\dots n\end{pmatrix},\quad 1\leqslant p<q<n.$$

Unfortunately this result requires still more; as a counterexample under the stated conditions we have for

$$A = \begin{bmatrix} 6 & 2 & 1 \\ 4 & 6 & 2 \\ 0 & 6 & 6 \end{bmatrix}, \qquad 720 = A\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} A\begin{pmatrix}2\\2\end{pmatrix} > A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} A\begin{pmatrix}2 & 3\\ 2 & 3\end{pmatrix} = 672.$$

This result (***) requires that (**) be true whenever $(i_1 \ldots i_s)$ and $(k_1 \ldots k_s)$ have $s-1$ elements in common (i.e., our condition (3)). If we assume that all principal minors of A are positive and (3) holds, their proof of (***) holds mutatis mutandis. Thus (3) implies (2) (both for $\alpha = \emptyset$ and $\alpha \neq \emptyset$); in the following theorem we summarize the above discussion and add a converse.

THEOREM 1. *Suppose A has all positive principal minors. Then* (2) *and* (3) *are equivalent.*

Proof. By our above remarks, we need only prove that (2) implies (3). We may assume without loss of generality that $\alpha = (1 \ldots m-2)$, $\beta = (m-1)$, $\gamma = (m)$. We define $D = [d_{ij}]$ by

$$d_{ij} = A\begin{pmatrix}\alpha & i\\ \alpha & j\end{pmatrix}, \qquad i, j = m-1, m.$$

Now by (2) and Sylvester's identity ([1], vol. I, p. 33),

$$\det D = A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} A\begin{pmatrix}\alpha & \beta & \gamma\\ \alpha & \beta & \gamma\end{pmatrix} \leqslant A\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} A\begin{pmatrix}\alpha & \gamma\\ \alpha & \gamma\end{pmatrix} = d_{m-1,m-1} d_{m,m},$$

so that

$$A\begin{pmatrix}\alpha & \beta\\ \alpha & \gamma\end{pmatrix} A\begin{pmatrix}\alpha & \gamma\\ \alpha & \beta\end{pmatrix} = d_{m-1,m} d_{m,m-1} \geqslant 0.$$

The proof is complete.

In general, (1) does not imply (2) and (3); consider

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

If A is non-singular and in one of the classes previously mentioned, then it is known (cf. [4] and [5]) that all principal minors of A are positive and the theorem applies. If A is singular and either positive semi-definite or an M-matrix, then for all $t > 0$, $A_t = A + tI$ is either positive definite or a non-singular M-matrix and by continuity we have (1) and (2) for A. If A is singular and totally non-negative, A_t is not necessarily totally non-negative, but (3) is still satisfied for A_t, and again by continuity (1) and (2) hold for A.

In general, if we assume only that the principal minors of A are non-negative, (3) does not imply (1) or (2); consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

3. We next discuss the case of equality in (1), generalizing the known result for positive definite (cf. [1], vol. I, p. 255) and totally non-negative ([2], p. 113) matrices. We need two easily-proved propositions regarding the reducibility of matrices (cf. [1], vol. II, p. 50) and a lemma of independent interest.

PROPOSITION 1. *If A is reducible, for each permutation $(i_1 \ldots i_m)$ of $(1 \ldots m)$, $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_m i_1} = 0$.*

PROPOSITION 2. *If A is irreducible, there exists a product $a_{j_1 j_2} a_{j_2 j_3} \ldots \ldots a_{j_n j_1} \neq 0$, where each j, $1 \leqslant j \leqslant m$, occurs at least once as an index (we may exclude elements a_{jj} from the product).*

Proof of Proposition 2. The existence of the non-zero product in the statement is equivalent to: given any i, for each j there exists $i_2, \ldots, i_k$ such that $a_{ii_2} a_{i_2 i_3} \ldots a_{i_k j} \neq 0$. To prove this, choose i. First, there exists an i_2 so that $a_{ii_2} \neq 0$; then there exists an i_3 so that either $a_{ii_3} \neq 0$ or $a_{ii_2} a_{i_2 i_3} \neq 0$. Continuing in this fashion, we can clearly "connect" i to any j with a non-zero product.

LEMMA. *Suppose A has all positive principal minors and satisfies (3) (i.e., is weakly sign-symmetric). If*

$$A\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix} = A\begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix} A\begin{pmatrix} \gamma \\ \gamma \end{pmatrix},$$

then

$$A\begin{pmatrix} \alpha & \gamma \\ \alpha & \gamma \end{pmatrix} = A\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} A\begin{pmatrix} \gamma \\ \gamma \end{pmatrix}.$$

Proof. It is sufficient to prove the Lemma for $\alpha = (1, \ldots, p-1)$, $\beta = (p)$, $\gamma = (p+1, \ldots, m)$, where $1 < p < m$. We define $D = [d_{ij}]$ by

$$d_{ij} = A\begin{pmatrix} \alpha & i \\ \alpha & j \end{pmatrix}, \quad i, j = p, p+1, \ldots, m.$$

Now again by Sylvester's identity, D has all positive minors, is weakly sign-symmetric, and

$$A\begin{pmatrix}\alpha\ \beta\ \gamma\\ \alpha\ \beta\ \gamma\end{pmatrix} = \frac{D\begin{pmatrix}\beta\ \gamma\\ \beta\ \gamma\end{pmatrix}}{A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}} \leqslant \frac{d_{pp}D\begin{pmatrix}\gamma\\ \gamma\end{pmatrix}}{A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}} = \frac{A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\alpha\ \gamma\\ \alpha\ \gamma\end{pmatrix}}{A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}}$$

$$\leqslant A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\gamma\\ \gamma\end{pmatrix},$$

which contains what we wanted to prove.

COROLLARY. *Suppose A has all positive principal minors and is weakly sign-symmetric. If*

$$A\begin{pmatrix}\alpha\ \beta\ \gamma\\ \alpha\ \beta\ \gamma\end{pmatrix} = A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\gamma\\ \gamma\end{pmatrix},$$

then

$$A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}A\begin{pmatrix}\alpha\ \beta\ \gamma\\ \alpha\ \beta\ \gamma\end{pmatrix} = A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\alpha\ \gamma\\ \alpha\ \gamma\end{pmatrix}.$$

Proof. The proof is simple: by the Lemma,

$$A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}A\begin{pmatrix}\alpha\ \beta\ \gamma\\ \alpha\ \beta\ \gamma\end{pmatrix} = A\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\gamma\\ \gamma\end{pmatrix} = A\begin{pmatrix}\alpha\ \beta\\ \alpha\ \beta\end{pmatrix}A\begin{pmatrix}\alpha\ \gamma\\ \alpha\ \gamma\end{pmatrix}.$$

THEOREM 2. *Suppose A has all positive principal minors and is weakly sign-symmetric. If equality occurs in* (1) *for some* $\{\beta, \gamma\}$, *then*

$$A\begin{pmatrix}\beta\ \gamma\\ \beta\ \gamma\end{pmatrix}$$

is reducible.

Proof. We may assume that $\beta = (1, \ldots, p)$, $\gamma = (p+1, \ldots, m)$, where $1 \leqslant p < m$. The proof is by induction on m. For $p = 1$, $m = 2$, $\det A = a_{11}a_{22}$ certainly implies that $a_{12}a_{21} = 0$ and A is reducible.

By our Lemma and the inductive hypothesis, all of the matrices

$$A\begin{bmatrix}i_1 \ldots i_k\\ i_1 \ldots i_k\end{bmatrix}, \qquad k < m,\ i_q \leqslant p,\ i_r > p \text{ for some } q, r,$$

are reducible, so that by Proposition 1, for all such $(i_1 \ldots i_k)$ we have

$$a_{i_1i_2} \ldots a_{i_ki_1} = 0. \tag{4}$$

By a *cyclic product* $C(a)$ we shall mean a product $a_{i_1 i_2} \ldots a_{i_k i_1}$, where $a = (i_1 \ldots i_k)$ is an ordered collection of distinct indices. Now the determinant of A is a sum of terms of the form $\pm a_{1j_1} \ldots a_{mj_m}$, and each term may be rearranged as a product of cyclic products $C(a_i)$, where the a_i are pairwise disjoint. If a term is a product of $C(a_i)$, each of which has all indices $i_q \leqslant p$ or all indices $i_q > p$, then the term appears in the expansion of the right-hand side of (1). If a term contains a $C(a_i)$ of the form (4), then the term is zero. The only other possibility is for the term to be a cyclic product with m indices (and factors).

We have therefore

$$0 = \det A - A\begin{pmatrix}\beta\\ \beta\end{pmatrix} A\begin{pmatrix}\gamma\\ \gamma\end{pmatrix} = \sum \pm a_{i_1 i_2} \ldots a_{i_m i_1}. \tag{5}$$

We wish to prove that each of these terms is zero. Suppose $a_{j_1 j_2} \ldots a_{j_m j_1} \neq 0$, with $j_1 \leqslant p, j_m > p$. Now $a_{j_1 j} = 0$ for $j \neq j_1, j_2$, or otherwise $a_{j_1 j} a_{j\, j+1} \ldots a_{j_m j_1} \neq 0$, contradicting (4). Similarly $a_{j_2 j} = 0$ if $j \neq j_1, j_2, j_3$. Continuing this process, we see that the only possible non--zero cyclic product with n factors is the one we have chosen. Then, however, by (5), we have a contradiction. Hence all cyclic products with n factors are zero.

Consider each product $a_{j_1 j_2} \ldots a_{j_n j_1}$ in which each j, $1 \leqslant j \leqslant m$, occurs as an index. This product is a product of $C(a_i)$, where the a_i are of course not pairwise disjoint. As each j occurs as an index, some $C(a_i)$ must contain indices $j_q \leqslant p, j_r > p$, and must be zero. Hence, by Proposition 2, A is reducible, and the proof is complete.

We note that equality in (1) does not imply

$$A\begin{bmatrix}\beta\\ \gamma\end{bmatrix} = 0 \quad \text{or} \quad A\begin{bmatrix}\gamma\\ \beta\end{bmatrix} = 0.$$

For example, take $\beta = (1\,2)$, $\gamma = (3)$ in

$$A = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}.$$

Also, equality in (2) for some $\{a \neq \emptyset, \beta, \gamma\}$ does not imply that

$$A\begin{bmatrix}a & \beta & \gamma\\ a & \beta & \gamma\end{bmatrix}$$

is reducible. For example, for any choice of $\{\alpha = (i), \beta = (j), \gamma = (k)\}$, we have equality in (2) for

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

4. We conclude with a remark on real 3×3 matrices.

THEOREM 3. *Suppose A is a real 3×3 matrix with all positive principal minors. If* (1) *is satisfied, then all characteristic roots of A have positive real parts.*

Proof. We first prove that A has no imaginary roots. Suppose A has roots λ, $\pm\mu i$ (λ, μ real). Then

$$\sum_i a_{ii} = \lambda > 0, \qquad \sum_{i<j} A\begin{pmatrix} i\ j \\ i\ j \end{pmatrix} = \mu^2 > 0, \qquad \det A = \lambda\mu^2 > 0.$$

From this we have

$$\det A = \lambda\mu^2 = \sum a_{ii} \sum A\begin{pmatrix} i\ j \\ i\ j \end{pmatrix} > a_{11} A\begin{pmatrix} 2\ 3 \\ 2\ 3 \end{pmatrix} \geqslant \det A,$$

which is impossible.

Let $A_t = A + tI$, $t > 0$. It is easy to see that A_t has all positive principal minors, and that (1) is satisfied; by above A_t has no imaginary roots $\pm\mu i$, so that A has no roots $-t\pm\mu i$ with negative real parts.

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OREGON STATE UNIVERSITY
CORVALLIS, OREGON, U. S. A.

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