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SOME NEW CONSTRUCTIONS OF 4-TUPLE SYSTEMS

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In 1852 Steiner [12] posed the following problem: For which integer N is it possible to form a system V_3 of triples of numbers $1, \ldots, N$ in such a way that every pair of numbers appears in exactly one triple? Assuming this is solved we can ask for the possibility of forming quadruples such that any triple not in V_3 should appear in exactly one quadruple, and that no quadruple should contain a triple from V_3. Does this impose a new condition on the number N? Steiner carries on stating analogous problems for quintuples, sixtuples, etc.

The generalized Steiner's problem is as follows: Given four positive integers $v > l > k, \lambda$ we consider the proposition $P(v, l, k, \lambda)$ meaning that for every set S having v elements there exists a system V of subsets of S having l elements each and such that every subset of S having k elements is contained in exactly λ sets of the system. We shall call V a *realization* of $P(v, l, k, \lambda)$. Sometimes it is also called *tactical configuration* (or, briefly, *configuration*). Configurations with $k = 2$ are known as *balanced incomplete block designs* (BIBD). The number of l-element subsets belonging to a realization of $P(v, l, k, \lambda)$ and containing some fixed h elements is equal to $\lambda\binom{v-h}{k-h}\Big/\binom{l-h}{k-h}$ whence $P(v, l, k, \lambda)$ implies that

$$(1) \qquad \lambda\binom{v-h}{k-h}\Big/\binom{l-h}{k-h} \text{ is an integer for } h = 0, 1, \ldots, k-1.$$

It is also known that in general proposition (1) does not imply proposition $P(v, l, k, \lambda)$. For instance if $l = 6$, $k = 2$ and $\lambda = 1$ (see [13]) or if $l = 5$, $k = 2$ and $\lambda = 2$ and also for some other BIBD (see [2], [8] and [10]) it does not. Nevertheless, for some l, k, and λ formula (1) does imply $P(v, l, k, \lambda)$, e.g. for $l = 3$, $k = 2$ and every λ ([6], [9] and [11]), for $l = 4$, $k = 2$ and every λ (see [4]), for $l = 5$, $k = 2$, $\lambda = 1, 4, 20$ (except possibly $v = 141$; see [4]), $l = 4$, $k = 3$ for every λ ([3] and [5]), etc. (see [1] and [14]).

This paper is concerned with $P(\nu, 4, 3, 1)$. In this case (1) takes the form $\nu \equiv 2$ or $4 \pmod 6$, and it is known [3] that this condition is sufficient. We want to present new construction yielding realizations different from those given in [3] for $l = 4$, $k = 3$, $\lambda = 1$ and

I. $\nu = mn+1$, where $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 1$ or $3 \pmod 6$,

II. $\nu = mn+2$, where $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 8 \pmod{12}$,

III. $\nu = mn+4$, where $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 10 \pmod{12}$,

IV. $\nu = mn+2$, where $m = 4^a$, and $n \equiv 0$ or $2 \pmod 6$.

There are known theorems of the following type: given a realization of $P(n, 4, 3, 1)$, a realization of $P\big(f(n), 4, 3, 1\big)$ is constructed; e.g. 1° a theorem of Witt [16] shows how to obtain a realization of $P(2n, 4, 3, 1)$ from a given $P(n, 4, 3, 1)$ and a realization of $P(mn, 4, 3, 1)$ from any given realizations of $P(n, 4, 3, 1)$ and $P(m, 4, 3, 1)$ and 2° a theorem of Hanani shows how to construct a realization of $P(12n+2, 4, 3, 1)$ from a realization of $P(n+1, 4, 3, 1)$. Using these theorems it is possible to construct different realizations of $P(n, 4, 3, 1)$ for the same n. We conjecture that in many cases these realizations are not isomorphic (i.e. they do not follow from each other by a simple renumbering of elements). In fact, we have checked that in the case of $\nu = 22$ the realization obtained by construction I below is not isomorphic to that given in [3].

Construction I. Let $m \equiv 1$ or $3 \pmod 6$, $n \equiv 1$ or $3 \pmod 6$, and $S = \{0\} \cup \{(i, j): i = 1, \ldots, m, j = 1, \ldots, n\}$. Define now:

$$A_i = \{(i, 1), (i, 2), \ldots, (i, n)\} \quad \text{for} \quad i = 1, \ldots, m;$$

$$B_j = \{(1, j), \ldots, (m, j)\} \quad \text{for} \quad j = 1, \ldots, n;$$

$$A = \{A_1, \ldots, A_m\}, \quad B = \{B_1, \ldots, B_n\},$$

$$A = A \cup \{1\},$$

$$B^* = B \cup \{0\}.$$

Let L_1 be a realization of $P(n+1, 4, 3, 1)$ in the set B^*, and for every i let L_1^i be a realization of the same proposition in the set $A_i \cup \{0\}$ resp. If we delete 0 from all 4-tuples belonging to L_1 that contain it, we shall get a set of triplets which gives a realization of $P(n, 3, 2, 1)$ in B. Let us call it L_2. Let L_3 be a realization of $P(m+1, 4, 3, 1)$ in the set A^*. If we delete 1 from all 4-tuples that contain it, we shall get a set of triplets which gives a realization of $P(n, 3, 2, 1)$ in A. Let us call it L_4. Let R_1 be the set of all 4-tuples of the form $\{(t, h), (u, i), (v, j), (w, k)\}$, where $\{A_t, A_u, A_v, A_w\} \in L_3$, and $h, i, j, k = 1, \ldots, n$ and $h+i+j+k \equiv 0 \pmod n$. Let R_2 be the set of all 4-tuples of the form $\{0, (x, q), (y, q), (z, q)\}$ or $\{0, (x, q), (y, r), (z, s)\}$, where $\{A_x, A_y, A_z\} \in L_4$, $\{B_q, B_r, B_s\} \in L_2$. Let R_3 be the set of 4-tuples: $\{(x, q), (x, r), (y, q), (y, s)\}$ or $\{(x, q), (x, r),$

$(z, q), (y, r)\}$, where x, y, z, q, r, s are subject to the same restrictions as in the construction of R_2, and let $(y-x)(z-x)(x-y) > 0$, $(r-q)(s-q)\times \times(s-r) > 0$. Let $\bar{R}_4$ be the set of all 4-tuples $\{(i, e), (i+k, f), (j, g), (j+ +k, h)\}$ in the set $A_1 \cup A_2 \cup A_3$, where $i, j = 1, 2, 3$, $k = 0, 1, 2$ (but for $j = i$, we take only $k \neq 0$), and addition is mod 3, $\{B_e, B_f, B_g, B_h\} \epsilon L_1$, $e < f < g < h$. For every $\tau = \{A_x, A_y, A_z\} \epsilon L_4$, let $\bar{R}_4^\tau$ denote $\bar{R}_4$ in the set $A_x \cup A_y \cup A_z$. Put $\bigcup_{\tau \epsilon L_4} \bar{R}_4^\tau = R_4$. We shall prove that *the set V of all 4-tuples of the set $R_1 \cup R_2 \cup R_3 \cup R_4 \cup L_1^1 \cup \ldots \cup L_1^m$ forms a realization of $P(mn+1, 4, 3, 1)$ in S.*

We show first that every triplet of elements of S is contained in at least one 4-tuple in the set V. Suppose that $\{a, b, c\}$ is such a triplet and $a = (\alpha, \mathfrak{a})$, $b = (\beta, \mathfrak{b})$, $c = (\gamma, \mathfrak{c})$. If α, β, γ are all equal i, say, then $\{a, b, c\}$ is contained in a 4-tuple of the system L_1^i. If $\alpha = \beta \neq \gamma$, then there exists a δ such that $\{A_\alpha, A_\gamma, A_\delta\} \epsilon L_4$. Now $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ cannot be all equal. If two of them are equal, say $\mathfrak{a} = \mathfrak{b} \pm \mathfrak{c}$, then there exists a $\mathfrak{d}$ such that $\{B_\mathfrak{a}, B_\mathfrak{c}, B_\mathfrak{d}\} \epsilon L_2$ and then $\{a, b, c, (\gamma, \mathfrak{d})\} \epsilon R_3$ if $(\gamma-\alpha)(\delta-\alpha)(\delta-\gamma)\times \times(\mathfrak{c}-\mathfrak{a})(\mathfrak{d}-\mathfrak{a})(\mathfrak{d}-\mathfrak{c}) > 0$ and $\{a, b, c, (\delta, \mathfrak{c})\} \epsilon R_3$ if $(\gamma-\alpha)(\delta-\alpha)(\delta-\gamma)(\mathfrak{c}-\mathfrak{a})$ $(\mathfrak{d}-\mathfrak{a})(\mathfrak{d}-\mathfrak{c}) < 0$. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are all different and $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}\} \epsilon L_2$, then $\{a, b, c, (\gamma, \mathfrak{a})\} \epsilon R_3$ if $(\gamma-\alpha)(\delta-\alpha)(\delta-\gamma)(\mathfrak{b}-\mathfrak{a})(\mathfrak{c}-\mathfrak{a})(\mathfrak{c}-\mathfrak{b}) > 0$ and $\{a, b, c, (\gamma, \mathfrak{b})\} \epsilon R_3$ if $(\gamma-\alpha)(\delta-\alpha)(\delta-\gamma)(\mathfrak{b}-\mathfrak{a})(\mathfrak{c}-\mathfrak{a})(\mathfrak{c}-\mathfrak{b}) < 0$. Finally, if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are all different and $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}\} \notin L_2$, then there exists a $\mathfrak{d}$ such that $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}, B_\mathfrak{d}\} \epsilon L_1$. Let $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\} = \{e, f, g, h\}$, where $e < f < g < h$. Then $\{a, b, c(\delta, \mathfrak{d})\} \epsilon R_4$ if $\{\mathfrak{a}, \mathfrak{b}\} = \{e, h\}$ or $\{\mathfrak{a}, \mathfrak{b}\} = \{f, g\}$, and $\{a, b, c, (\gamma, \mathfrak{d})\} \epsilon R_4$ if $\{\mathfrak{a}, \mathfrak{b}\} = \{e, f\}$ or $\{\mathfrak{a}, \mathfrak{b}\} = \{g, h\}$ or $\{\mathfrak{a}, \mathfrak{b}\} = \{e, g\}$ or $\{\mathfrak{a}, \mathfrak{b}\} = \{f, h\}$. It remains to consider the case where α, β, γ are all different. If $\{A_\alpha, A_\beta, A_\gamma\} \notin L_4$, then there exists a δ such that $\{A_\alpha, A_\beta, A_\gamma, A_\delta\} \epsilon L_3$. Then $\{a, b, c, (\delta, -\mathfrak{a}-\mathfrak{b}-\mathfrak{c})\} \epsilon R_1$ where the addition is understood mod n. Suppose that $\{A_\alpha, A_\beta, A_\gamma\} \epsilon L_4$. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are all equal, then $\{a, b, c, 0\} \epsilon R_2$. If $\mathfrak{a} = \mathfrak{b} \neq \mathfrak{c}$, then there exists a $\mathfrak{d}$ such that $\{B_\mathfrak{a}, B_\mathfrak{c}, B_\mathfrak{d}\} \epsilon L_2$ and we have $\{a, b, c, (\beta, \mathfrak{c})\} \epsilon R_3$ for $(\beta-\alpha)(\gamma-\alpha)\times (\gamma-\beta)(\mathfrak{c}-\mathfrak{a})(\mathfrak{d}-\mathfrak{a})(\mathfrak{d}-\mathfrak{c}) > 0$ and $\{a, b, c, (\alpha, \mathfrak{c})\} \epsilon R_3$ for $(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)\times \times(\mathfrak{c}-\mathfrak{a})(\mathfrak{d}-\mathfrak{a})(\mathfrak{d}-\mathfrak{c}) < 0$. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are all different and $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}\} \epsilon L_2$, then $\{a, b, c, 0\} \epsilon R_2$. Finally, if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are all different and $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}\} \notin L_2$, then there exists a $\mathfrak{d}$ such that $\{B_\mathfrak{a}, B_\mathfrak{b}, B_\mathfrak{c}, B_\mathfrak{d}\} \epsilon L_1$. Let $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\} = \{e, f, g, h\}$, where $e < f < g < h$. Then $\{a, b, c, (\alpha, \mathfrak{d})\} \epsilon R_4$, $\{\mathfrak{a}, \mathfrak{d}\} = \{e, h\}$ or $\{f, g)$, $\{a, b, c, (\beta, \mathfrak{d})\} \epsilon R_4$, if $\{\mathfrak{b}, \mathfrak{d}\} = \{e, h\}$ or $\{f, g\} \{a, b, c, (\gamma, \mathfrak{d})\} \epsilon R_4$ if $\{\mathfrak{c}, \mathfrak{d}\} = \{e, h\}$ or $\{f, g\}$.

In the case of $a = 0$ it is sufficient to show that every pair of elements of S is contained in a 4-tuple in which the element 0 is contained as well. Every pair whose elements belong to the same A_i is evidently contained in a 4-tuple of the system L_1^i, and the remaining pairs are contained in 4-tuples of the system R_2.

To prove that our system is a realization of $P(mn+1, 4, 3, 1)$ it is enough to show that every triplet of elements of S is contained in at most one 4-tuple from our system. This will be clear if we show that $|V| \leqslant \binom{|S|}{3}\Big/\binom{4}{3}$. Since $\binom{|S|}{3}\Big/\binom{4}{3} = (mn+1)mn(mn-1)/24$, it is sufficient to show that

$$|V| = \frac{(mn+1)mn(mn-1)}{24}. \tag{2}$$

We have $|R_1| = n^3 m(m-1)(m-3)/24$, which is seen by subtracting the number of 4-tuples in the realization $P(m+1, 4, 3, 1)$ with one element fixed from the number of all 4-tuples in this realization, and by multiplying the obtained number by n^3 (as i, j, k run independently over n values). We have next $|R_2| = n^2 m(m-1)/6$, because there are $m(m-1)/6$ triplets in L_4 and with every such triple n values of q can be associated. Every such q should be adjoined to the $n-1$ pairs (r, s) and to the pair (q, q). Further, we have $|R_3| = |L_4|\cdot|L_2|\cdot 3^2\cdot 2 = n(n-1)m\times\times(m-1)/2$. The factor 3 occurs here as the number of those permutations of three numbers, say x, y, z, which do not change the inequality $(y-z)\times\times(z-x)(z-y) > 0$. The factor 2 corresponds to the fact that in the definition of R_3 there are two formulae. Now we have $|R_4| = n(n-1)(n-3)m\times\times(m-1)/6$, which is seen by subtracting the number of 4-tuples in the realization $P(n+1, 4, 3, 1)$ with one element fixed form the number of all 4-tuples. The obtained number is to be multiplied by $3^3-3(=24)$ (i, j, k run here independently over 3 values, except $j = i$, $k = 0$), and by $m(m-1)/6$ (i.e. the number of triplets in L_4). Finally, we see that

$$\Big|\bigcup_{i=1}^{m} L_1^i\Big| = \frac{(n+1)n(n-1)m}{24}.$$

Consequently,

$$|V| = \Big|R_1 \cup R_2 \cup R_3 \cup R_4 \cup \bigcup_{i=1}^{m} L_1^i\Big| = \frac{(nm+1)nm(nm-1)}{24},$$

which implies (2) and thus completes the proof.

Construction II. Let S be the set consisting of $0, 1$ and all pairs (i, j), where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $A_i = \{(i, 1), \ldots, (i, n)\} \cup \{0, 1\}$ $A = \{A_1, \ldots, A_m\}$, and $A^* = A \cup \{2\}$. Let P_1 be a realization of $P(m+1, 4, 3, 1)$ in the set A^*, and for every i let P_2^i be a realization of $P(n+2, 4, 3, 1)$ in the set A_i. If we delete 2 from all 4-tuples, which contain it, we shall get a set of triplets which gives a realization of $P(m, 3, 2, 1)$ in A. Let us call it P_3. Hanani [3] has constructed a realization E of $P(3n+2, 4, 3, 1)$ in $A_1 \cup A_2 \cup A_3$ such that $\bigcup_{i=1}^{3} P_2^i \subset E$. For every triplet $t = \{A_x, A_y, A_z\} \in P_3$ we construct an analogous realization E^t of $P(3n+2, 4, 3, 1)$ in $A_x \cup A_y \cup A_z$. Clearly, $P_2^x \cup P_2^y \cup P_2^z \subset E^t$. Put $L^t = E^t \setminus (P_2^x \cup P_2^y \cup P_2^z)$. Finally, we construct the set R_0 of all 4-tuples of the form $\{(t, i), (u, i+k), (v, j), (w, j+k)\}$, where $i, j = 1, \ldots, n$ $k = 0, 1, \ldots, n-1$, the addition is to be understood as addition mod n, and $\{A_t, A_u, A_v, A_w\} \in P_1$, $t < u < v < w$. We claim that *the set*

$$V = \bigcup_{i=1}^{m} P_2^i \cup \bigcup_{t \in P_3} L^t \cup R_0$$

is a realization of $P(mn+2, 4, 3, 1)$ *in* S. At first we prove that every triplet of elements of S is a subset of a 4-tuple in the system V. Let $\{a, b, c\}$ be such a triplet. If for some i we have $a, b, c \in A_i$, then $\{a, b, c\}$ is contained in a 4-tuple from P_2^i. If $a, b \in A_i$, $c \in A_j$ $(i \neq j)$, then there exists a z such that $t = \{A_i, A_j, A_z\} \in P_3$ and $\{a, b, c\}$ is contained in a 4-tuple of L^t. If $a \in A_i$, $b \in A_j$, $c \in A_k$ (i, j, k are all different), then in the case of $\{A_i, A_j, A_k\} = t \in P_3$ we see that $\{a, b, c\}$ is contained in a 4-tuple of L^t and in the other case $\{a, b, c\}$ is contained in a 4-tuple of R_0. In order to prove that every triplet from S is contained in exactly one 4-tuple from the system V, it is sufficient to show that the number of elements of the system V is not larger than $\binom{|S|}{3} \Big/ \binom{4}{3}$.

We have

$$\Big| \bigcup_{i=1}^{m} P_2^i \Big| = \frac{(n+2)(n+1)nm}{24},$$

which is the number of 4-tuples in m realizations of $P(n+2, 4, 3, 1)$. Further, $|R_0| = n^3 m(m-1)(m-3)/24$ because we must subtract the number of 4-tuples in the realization of $P(m+2, 4, 3, 1)$ with one element fixed from that of all 4-tuples in this realization, and multiply the number so obtained by n^3 (i, j, k run independently over n values). We have also

$$|L^t| = |E^t| - |P_2^x \cup P_2^y \cup P_2^z| = \frac{4n^3+3n^2}{4},$$

because we must subtract the number of all 4-tuples in the realization of $P(n+2, 4, 3, 1)$ multiplied by 3 from the number of all 4-tuples in the realization of $P(3n+2, 4, 3, 1)$.

Finally,

$$|V| = \Big| \bigcup_{i=1}^{m} P_2^i \cup R_0 \cup \bigcup_{t \in P_3} L^t \Big| = \frac{(mn+2)(mn+1)mn}{24}.$$

On the other hand,

$$\binom{|S|}{3}\Big/\binom{4}{3} = \frac{(mn+2)(mn+1)mn}{24},$$

which completes the proof.

Construction III. Let S be the set consisting of $0, 1, 2, 3$ and all pairs (i, j), where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Let

$$A_i = \{(i, j), \ldots, (i, n)\} \cup \{0, 1, 2, 3\}, \quad A = \{A_1, \ldots, A_m\},$$
$$A^* = A \cup \{4\}.$$

Let P_1 be a realization of $P(m+1, 4, 3, 1)$ in the set A^* and, for every i, let $\overline{P}_4^i$ be a realization of $P(n+4, 4, 3, 1)$ in A_i. Let $P_4^i = \overline{P}_4^i \setminus \{\{0, 1, 2, 3\}\}$. If we delete 4 from all 4-tuples that contain it, we shall get a set of triplets forming a realization of $P(m, 3, 2, 1)$ in A. Let us call it P_5. Hanani [3] has constructed a realization E of $P(3n+4, 4, 3, 1)$ in the set $A_1 \cup A_2 \cup A_3$ such that $\bigcup_{i=1}^{3} P_4^i \subset E$. We construct an analogous realization E^t of $P(3n+4, 4, 3, 1)$ in $A_x \cup A_y \cup A_z$ for every triplet $t = \{A_x, A_y, A_z\} \in P_5$. Clearly, $P_4^x \cup P_4^y \cup P_4^z \subset E^t$. Put $L^t = E^t \setminus (P_4^x \cup P_4^y \cup P_4^z) \setminus \{\{0, 1, 2, 3\}\}$. Let R_5 be the set of all 4-tuples of the form

$$\{(t, i), (u, i+k), (v, j), (w, j+k)\},$$

where $i, j = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, the addition is to be understood mod n, $\{A_t, A_u, A_v, A_w\} \in P_1$ and $t < u < v < w$. We claim that, *the set*

$$V = \bigcup_{i=1}^{m} P_4^i \cup \bigcup_{t \in P_5} L^t \cup R_5 \cup \{\{0, 1, 2, 3\}\}$$

is a realization of $P(mn+4, 4, 3, 1)$ *in* S.

At first we prove that every triplet of elements of S is contained in a 4-tuple from V. Let $\{a, b, c\}$ be such a triplet. If $a, b, c \in A_i$ with suitable i, then this triplet is contained in a 4-tuple from P_4^i, or in $\{0, 1, 2, 3\}$. If $a, b \in A_i$, $c \in A_j$ $(i \neq j)$, then there exists a z such that $\{A_i, A_j, A_z\} = t \in P_5$ and $\{a, b, c\}$ is contained in a 4-tuple from L^t. And if $a \in A_i$, $b \in A_j$, $c \in A_k$ $(i, j, k$ are all different), then in the case of $\{A_i, A_j, A_k\} = t \in P_5$ the triplet $\{a, b, c\}$ is contained in a 4-tuple from L^t, and in the case of $\{A_i, A_j, A_k\} \notin P_5$ the triple $\{a, b, c\}$ is contained, in a 4-tuple from R_5. To prove that every triplet of elements of S is contained in exactly one 4-tuple from V it is sufficient to show that the number of elements of the system V is not larger than $\binom{|S|}{3}\Big/\binom{4}{3}$.

We have

$$|\bigcup_{i=1}^{m} P_4^i| = \left(\frac{(n+4)(n+3)(n+2)}{24} - 1\right)m,$$

because we must subtract $|\{\{0, 1, 2, 3\}\}|$ from the number of all 4-tuples in the realization $P(n+4, 4, 3, 1)$ and then multiply the remainder by $|A|$ $(= m)$. Further, $|R_5| = n^3 m(m-1)(m-3)/24$, since we must subtract the number of all 4-tuples in the realization of $P(m+4, 4, 3, 1)$ with one element fixed from that of all 4-tuples in this realization, then multiply the obtained number by n^3 $(i, j, k$ run independently over n values). Now,

$$|L^t| = |E^t| - |P_x^4 \cup P_y^4 \cup P_z^4| - |\{\{0, 1, 2, 3\}\}|$$

$$= \binom{3n+4}{3}\Big/\binom{4}{3} - 3\left(\binom{n+4}{3}\Big/\binom{4}{3} - 1\right) - 1 = \frac{4n^3+9n^2}{4},$$

$$|\bigcup_{t\epsilon P_5} L^t| = \frac{n^2(4n+9)m(m-1)}{24},$$

$$|V| = |\bigcup_{i=1}^{m} P_4^i \cup R_5 \cup \bigcup_{t\epsilon P_5} L^t \cup \{\{0, 1, 2, 3\}\}| = \frac{(mn+4)(mn+3)(mn+2)}{24}.$$

In the other hand,

$$\binom{|S|}{3}\Big/\binom{4}{3} = \frac{(mn+4)(mn+3)(mn+2)}{24},$$

which completes the proof.

Construction IV. Before we turn to the contruction itself let us show the realization P_6 of a $P(4^a, 4, 3, 1)$ and the realization P_7 of a $P(4^a, 4, 2, 1)$ such that $P_7 \subset P_6$.

Consider the set Z of all 4-tuples $\{a, b, c, d\}$ satisfying the conditions

$$a = \sum_{j=0}^{2a-1} \varepsilon_j(a)2^j, \quad b = \sum_{j=0}^{2a-1} \varepsilon_j(b)2^j, \quad c = \sum_{j=0}^{2a-1} \varepsilon_j(c)2^j, \quad d = \sum_{j=0}^{2a-1} \varepsilon_j(d)2^j,$$

where $\varepsilon_j(a), \varepsilon_j(b), \varepsilon_j(c), \varepsilon_j(d) = 0$ or 1 and

(3) $$\varepsilon_j(a)+\varepsilon_j(b)+\varepsilon_j(c)+\varepsilon_j(d) \equiv 0 \pmod 2.$$

The set Z is a realization of $P(4^a, 4, 3, 1)$, because if we take an arbitrary triplet $\{a, b, c\}$ from the set $\{0, 1, \ldots, 4^a-1\}$, we can find in this set a number d such that (3) will be satisfied. Moreover, such a d is unique. In fact, if

$$\varepsilon_j(a)+\varepsilon_j(b)+\varepsilon_j(c) \equiv 0 \pmod 2,$$

then $\varepsilon_j(d) = 0$, and if

$$\varepsilon_j(a)+\varepsilon_j(b)+\varepsilon_j(c) \equiv 1 \pmod 2,$$

then $\varepsilon_j(d) = 1$. Since $a \neq b \neq c \neq a$, the number d is different from a, b and c. For suppose $d = a$. It follows from (3) that $\varepsilon_j(b)+\varepsilon_j(c) \equiv 0 \pmod 2$ for every j, thus $b = c$, against the assumption.

Now take the set W of all 4-tuplets $\{k, l, m, n\}$ satisfying the conditions

$$0 \leqslant k < l < m < n < 4^a, \qquad k = \sum_{i=0}^{a-1} \eta_i(k) 4^i,$$

$$l = \sum_{i=0}^{a-1} \eta_i(l) 4^i, \qquad m = \sum_{i=0}^{a-1} \eta_i(m) 4^i, \qquad n = \sum_{i=0}^{a-1} \eta_i(n) 4^i,$$

where $\eta_i(k), \eta_i(l), \eta_i(m), \eta_i(n) = 0$, or 1, or 2, or 3 and

(4) for every i, the terms of the sequence $\big(\eta_i(k), \eta_i(l), \eta_i(m), \eta_i(n)\big)$ are all equal or form an even permutation of $(0, 1, 2, 3)$.

The set W is a realization of $P(4^a, 4, 2, 1)$, because if we take any two elements p, r $(p \neq r)$ from $\{0, 1, \ldots, 4^a-1\}$, then we can find a unique set $\{s, t\}$ such that $\{p, r, s, t\} \epsilon W$. Let β be the greatest number i such that $v = \eta_i(p) \neq \eta_i(r) = w$. We choose numbers x and y such that $\{v, w, x, y\} = \{0, 1, 2, 3\}$, and next define s and t as follows: If $\eta_i(p) = \eta_i(r)$, then $\eta_i(t) = \eta_i(s) = \eta_i(p)$. If $\eta_i(p) \neq \eta_i(r)$, then $\big(\eta_i(p), \eta_i(r), \eta_i(s), \eta_i(t)\big)$ is an even permutation of (v, w, x, y). Since $p \neq r$, β does exist, and since all numbers v, w, x, y, are different, p, r, s, t are all different too. Since we know first two terms of the sequence $\big(\eta_i(p), \eta_i(r), \eta_i(s), \eta_i(t)\big)$ and the whole sequence is an even permutation of (v, w, x, y), the order of its terms is uniquely determined and so the set $\{s, t\}$ is well determined too. Let $\{p, r, s, t\} = \{k, l, m, n\}$, where $k < l < m < n$ and then $\eta_\beta(k) = 0$, $\eta_\beta(l) = 1$, $\eta_\beta(m) = 2$, $\eta_\beta(n) = 3$. Since the terms of the sequence $\big(\eta_i(p), \eta_i(r), \eta_i(s), \eta_i(t)\big)$ are all equal or form an even permutation of (v, w, x, y), the terms of the sequence $\big(\eta_i(k), \eta_i(l), \eta_i(m), \eta_i(n)\big)$ are all equal or, as it is easy to verify, form an even permutation of $(0, 1, 2, 3)$ and (4) is satisfied.

Suppose that $\{p, r, s', t'\} \epsilon W$, where $\{s, t\} \neq \{s', t'\}$, and let $\{p, r, s', t'\} = \{k', l', m', n'\}$, where $k' < l' < m' < n'$. Then there exist indices i and j such that

$$\eta_i(p) \neq \eta_i(r), \qquad \eta_i(s) = \eta_i(s'), \qquad \eta_i(t) = \eta_i(t'),$$

$$\eta_j(p) \neq \eta_j(r), \qquad \eta_j(s) = \eta_j(t'), \qquad \eta_j(t) = \eta_j(s').$$

Since $\big(\eta_i(p), \eta_i(r), \eta_i(s), \eta_i(t)\big)$ and $\big(\eta_j(p), \eta_j(r), \eta_j(s), \eta_j(t)\big)$ are both even permutations of (v, w, x, y), $\big(\eta_j(p), \eta_j(r), \eta_j(s'), \eta_j(t')\big)$ is an odd permutation of $(\eta_i(p), \eta_i(r), \eta_i(s'), \eta_i(t'))$. Thus $\sigma_j = (\eta_j(k'), \eta_j(l'), \eta_j(m'), \eta_j(n'))$ is an odd permutation of $\sigma_i = (\eta_i(k'), \eta_i(l'), \eta_i(m'), \eta_i(n'))$ and either σ_j or σ_i is an odd permutation of $(0, 1, 2, 3)$, contrary to (4).

It remains to prove that W is a subset of Z. Let $\{p, r, s, t\} \in W$ and consider $\varepsilon_j(p), \varepsilon_j(r), \varepsilon_j(s), \varepsilon_j(t)$ for any $j < 2^a$. Let $j = 2i$ or $2i+1$. If $\eta_i(p) = \eta_i(r) = \eta_i(s) = \eta_i(t)$, then $\varepsilon_j(p) = \varepsilon_j(r) = \varepsilon_j(s) = \varepsilon_j(t)$. If $(\eta_i(p), \eta_i(r), \eta_i(s), \eta_i(t))$ is an even permutation of $(0, 1, 2, 3)$, then $(\varepsilon_j(p), \varepsilon_j(r), \varepsilon_j(s), \varepsilon_j(t))$ is a permutation of $(0, 0, 1, 1)$. In both cases $\varepsilon_j(p)+\varepsilon_j(r)+\varepsilon_j(s)+\varepsilon_j(t) \equiv 0 \pmod 2$ and so $\{p, r, s, t\} \in Z$.

Let S be the set consisting of 0,1 and all pairs (i, j), where $i = 1, \ldots, m, j = 1, \ldots, n$. Let $A_i = \{(i, 1), \ldots, (i, n)\} \cup \{0, 1\}$. $A = \{A_1, \ldots, A_m\}$. We shall construct in the set A a system P_6 which is a realization of $P(m, 4, 3, 1)$ and a system P_7 which is a realization of $P(m, 4, 2, 1)$ in such a way that P_7 will be a subset of P_6. For every i, let P_8^i be a realization of $P(n+2, 4, 3, 1)$ in the set A_i. In [3] there is a realization E of $P(4n+2, 4, 3, 1)$ in $A_1 \cup A_2 \cup A_3 \cup A_4$ such that $\bigcup_{i=1}^{4} P_8^i \subset E$. For every 4-tuple $t = \{A_x, A_y, A_z, A_w\} \in P_7$ we construct an analogous realization E^t of $P(4n+2, 4, 3, 1)$ in $A_x \cup A_y \cup A_z \cup A_w$. Clearly $P_8^x \cup P_8^y \cup P_8^z \cup P_8^w \subset E^t$. Put $L^t = E^t \setminus (P_8^x \cup P_8^y \cup P_8^z \cup P_8^w)$. Let R_6 be the set of all 4-tuples of the form $\{(p, i), (q, i+k), (r, j), (s, j+k)\}$, where $\{A_p, A_q, A_r, A_s\} \in P_6 \setminus P_7, p < q < r < s$, and $i, j = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, the addition being understood mod n. We claim that *the system*

$$V = \bigcup_{i=1}^{m} P_8^i \cup \bigcup_{t \in P_7} L^t \cup R_6$$

is a realization of $P(mn+2, 4, 3, 1)$ *in* S.

At first we prove that every triplet from S is contained in a 4-tuple from V. Let $\{a, b, c\}$ be such a triplet. If $a, b, c \in A_i$, then it is contained in a 4-tuple from P_8^i. If $a, b \in A_i$, $c \in A_j$ $(i \neq j)$, then there exist z, w such that $\{A_i, A_j, A_z, A_w\} = t \in P_7$ and so $\{a, b, c\}$ is contained in a 4-tuple from L^t. If $a \in A_i, b \in A_j, c \in A_k$ (i, j, k are all different), and $\{A_i, A_j, A_k\}$ is a subset of a 4-tuple t from P_7, then $\{a, b, c\}$ is contained in a 4-tuple from L^t and if not, then for a suitable z we have $\{A_i, A_j, A_k, A_z\} \in P_6 \setminus P_7$ and so $\{a, b, c\}$ is contained in a 4-tuple from R_6.

To prove that every triplet from S is contained in exactly one 4-tuple from V it is sufficient to show that the number of elements of the system V is not larger than $\binom{|S|}{3} \Big/ \binom{4}{3}$. Now

$$\Big|\bigcup_{i=1}^{m} P_8^i\Big| = \frac{(n+2)(n+1)nm}{24}$$

(the number of all 4-tuples of the realization $P(n+2, 4, 3, 1)$ multiplied by $|A|$ $(= m)$). $|R_6| = n^3 m(m-1)(m-4)/24 = (|P_6|-|P_7|)n^3$ (i, j, k run independently over n values). We further have

$$|L^t| = |E^t| - |P_8^x \cup P_8^y \cup P_8^z \cup P_8^w| = \binom{4n+2}{3}\Big/\binom{4}{3} - 4\binom{n+2}{3}\Big/\binom{4}{3} = \frac{n^2(5n+3)}{2}$$

and

$$\Big|\bigcup_{t\in P7} L^t\Big| = \frac{n^2(5n+3)m(m-1)}{24}.$$

Hence

$$|V| = \Big|R_6 \cup \bigcup_{i=1}^{m} P_8^i \cup \bigcup_{t\in P_7} L^t\Big| = \frac{(mn+2)(mn+1)mn}{24}.$$

On the other hand,

$$\binom{|S|}{3}\Big/\binom{4}{3} = \frac{(mn+2)(mn+1)mn}{24}$$

as well, which completes the proof.

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