COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 1

ON INSCRIBING AND CIRCUMSCRIBING HEXAGONS

BY

J. CEDER (SANTA BARBARA, CALIF.) AND B. GRÜNBAUM (JERUSALEM)

It is well-known that any closed convex curve has both inscribed and circumscribed centrally-symmetric hexagons (see [2], p. 242). In [1] it was shown that for a rotund and smooth convex curve C the set V of possible vertices of centrally-symmetric inscribed hexagons is dense in C. In this note we improve this result by showing, in particular, that when C is rotund and smooth V is, in fact, all of C. We shall also prove an analogous result for circumscribed centrally-symmetric hexagons.

The closed convex curve C is called *rotund* if it contains no line segments and is called *smooth* if each point has a unique line of support. Furthermore, we call a point $x \in C$ *wedgeless* if there is only one point on C which has a line of support, not passing through x, which is parallel to some line of support at x. A point $x \in C$ will be called *non-flat* if x is not interior to any line segment in C. Finally let H be the set of possible intersection points of C with circumscribed centrally-symmetric hexagons. Then our results may be stated as

THEOREM. *Let x be any wedgeless point on a closed convex curve C. Then*

(1) *There is an inscribed centrally-symmetric hexagon having vertex x.*

(2) *If x is also a non-flat point, there is a circumscribed centrally-symmetric hexagon tangent at x.*

In particular, then, when C is rotund and smooth we will have $C = V = H$. In general, however, neither V nor H need be all of C. For example, for the upper-half of the unit circle in the complex plane we have $H = C - \{i\} - (-1, +1)$ [1] and $V = C - \{-1, +1, i\}$. And for a triangle with vertices a, b, c we have $H = \{a, b, c\}$ and $V = C - \{a, b, c\}$.

Moreover, the conditions in the Theorem do not yield characterizations of V and H as one can easily construct examples where V contains a "wedge" point and H contains a "flat" point. One unsolved question

---

[1] (a, b) denotes the open interval with end points a, b.

is then: are there reasonable non-trivial characterizations of V and H **(P 581)** ? Another question would be: what is the class of curves for which $V = C$ or $H = C$? **(P 582)**

The proof of the Theorem. Let 0 be any wedgeless point on C, 0^* its unique antipodal point and L and L^* be parallel support lines at 0 and 0^* respectively. Moreover, choose the coordinate system so that 0 is the origin, L is the line $y = 0$ and L^* is the line $y = 1$. Let A be the open arc on C measured counter-clockwise from 0 to 0^*. Let us regard A also as a real open interval (a, b).

Proof of (1). For $x \in A$ let K_x consist of all midpoints of all inscribed parallelograms having 0 and x as vertices. Then K_x is a subset of the convex curve $(1/2)\ C$ and K_x is clearly either a singleton set or a closed line segment. Now let $k(x)$ be the directed distance along the curve $(1/2)\ C$ from the right hand end point of K_x to $(1/2)\ 0^*$.

First it is clear that there cannot exist a sequence $x(n) \to b$ for which each $K_{x(n)}$ is a non-degenerate closed interval. Hence, in some neighborhood of b, K_x will always be a singleton set. This in turn implies that k is continuous in some neighborhood of b. Also since 0 is a wedgeless point it is immediate that $\lim\limits_{x\to a} k(x) = \lim\limits_{x \to b} k(x) =$ zero. From these facts it follows that there exists a horizontal line hitting the graph of k in two distinct points, say, $\langle v, k(w)\rangle$ and $\langle w, k(w)\rangle$, where, moreover, v and w can be taken arbitrarily close to a and b respectively. This yields two distinct inscribed parallelograms having the same center and having 0 as a vertex. By the choice of v and w and the fact that 0 is a wedgeless point, the convex hull of the union of these two parallelograms will be a non-degenerate hexagon, which will clearly be the desired inscribed centrally-symmetric hexagon having 0 as a vertex. (Incidentally since there were uncountably many choices possible for v and w it follows that there are uncountably many such hexagons having 0 as a vertex.)

Proof of (2). Let 0^* be a non-flat point and assume that C is not centrally-symmetric, otherwise the proof is trivial. Let M be the line $y = \frac{1}{2}$. For $x \in A$ let $[f_L(x), f_R(x)]$ be the closed interval on the line L, formed by intersecting all lines of support at x with L. Likewise let $[g_L(x), g_R(x)]$ be the closed interval on L^* (coordinatized in the obvious way) formed by intersecting with L^* all the "opposite" support lines parallel to (and distinct from) the support lines at x. Finally let $K_x = [k_L(x), k_R(x)]$ be the closed interval on M formed by intersecting M with the closed trapezoid determined by the points corresponding to $f_L(x), f_R(x)$, $g_L(x)$ and $g_R(x)$.

In case C is rotund and smooth K_x is always a singleton set and yields a continuous function of $x \in A$, which fact easily leads to the desired result. In the general case we proceed as follows:

Since f_L, f_R, g_L and g_R are monotonic functions and $k_R = \frac{1}{2}|g_R - f_R|$ and $k_L = \frac{1}{2}|g_L - f_L|$, we see that k_R and k_L will be of bounded variation on each closed subinterval of A and thus they will have only jump discontinuities, countably many at most. Also since A can have only countably many corners, we have $k_L = k_R$ except on a countable set. Therefore it follows that $k_R(x) = k_L(x+) = k_R(x+)$ and $k_L(x) = k_L(x-) = k_R(x-)$ for all $x \in A$.

Now let K be the graph of k_R(or k_L) with the "jumps" filled in by vertical lines. That is, put K equal to the union of the line segments joining the points $\langle x, k_R(x-)\rangle$ and $\langle x, k_R(x+)\rangle$ as x ranges over all of A.

Now we choose $v \in A$ such that (i) $k_R(v) = k_L(v)$; (ii) v belongs to no non-degenerate subinterval I of A for which $k_R(u) = k_L(u) = k_R(v)$ for all $u \in I$; and (iii) there are points of K above and below the line $y = k_R(v)$. In fact, because C is not centrally-symmetric and because of the above properties of k_R and k_L, there will be uncountably many such points v.

Because 0 is a wedgeless point we have $\lim\limits_{x\to a} k_R(x) = \lim\limits_{x\to b} k_R(x) = \frac{1}{2}$ the first coordinate of 0^* (similarly for k_L). Moreover K is clearly a connected set. Hence, the line $y = k_R(v)$ must intersect K at least twice. So let $w \neq v$ be such that $\langle w, k_R(v)\rangle \in K$. This means that $K_v \cap K_w \neq \emptyset$. Then, letting $z \in K_v \cap K_w$, there are, according to construction, lines of support L_1 and L_2 at v and w respectively and corresponding opposite lines of support L_1^* and L_2^* such that z is the midpoint of the line segments joining $L \cap L_i$ and $L^* \cap L_i^*$ for $i = 1, 2$. Because of the choice of v it follows that the six lines L, L_1, L_2, L^*, L_1^* and L_2^* are all distinct. Finally, it easily follows that they will form the desired centrally-symmetric hexagon circumscribing C and tangent at 0. (Again because of the range of choices for v there will be uncountably many such hexagons tangent at 0.)

REFERENCES

[1] J. Ceder, *A property of planar convex bodies*, Israel Journal of Mathematics 1.4 (1963), p. 248-253.

[2] B. Grünbaum, *Measures of symmetry for convex sets*, Proceedings of Symposia in Pure Mathematics of American Mathematical Society 7 (1963), p. 233-270.

UNIVERSITY OF CALIFORNIA AT SANTA BARBARA
HEBREW UNIVERSITY, JERUSALEM

Reçu par la Rédaction le 21. 10. 1965