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A CLASSIFICATION OF CONTINUA AND WEAKLY CONFLUENT MAPPINGS

BY

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This note* contains a partial answer to a question, raised by J. B. Fugate, asking for a characterization of those continua which are the weakly confluent images of dendrites**.

A continuum X belongs to class $\mathscr{A}$ is understood to mean that each connected subset of X is path-connected. A curve X is *regular* provided that X has a basis of open sets each having finite boundary (see [5], p. 275). It is known that every curve in class $\mathscr{A}$ is regular (see [11], p. 323) and every regular curve is hereditarily locally connected (see [5], p. 283). We understand that a mapping f from a continuum X onto a continuum Y is *weakly confluent* means that if K is a continuum in Y, then some component of $f^{-1}(K)$ is mapped onto all of K by f. Observe that each retraction is a weakly confluent mapping.

G is a *graph* is understood to mean that there is a finite collection of arcs $\{a_i\}_{i=1}^n$ such that

$$G = \bigcup_{i=1}^{n} a_i$$

and such that if $x \in a_i \cap a_j$, $i \neq j$, then x is an endpoint of both a_i and a_j. T is a *finite tree* means that T is an acyclic graph.

We say that a continuum X is *strongly regular* provided that there is a sequence of finite sets $\{S_n\}_{n=1}^\infty$ such that if n is a positive integer, then $S_n \subset X$ and $X \setminus S_n$ has just finitely many components and each of these components has diameter less than $1/n$. It is immediate that every strongly regular curve is a regular curve.

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EXAMPLE 1. *There is a regular curve which is not strongly regular.*

Proof. Let X be the triangular Sierpiński curve (see [5], p. 276). Let $E = \{v_1, v_2, \ldots\}$ be the collection of vertices of the triangles defined in the construction of X.

Disjoint from X, let M be the union of all straight segments in the plane with one endpoint $(0, 0)$ and the other endpoint in

$$\left\{(r, \theta)\colon \theta = \frac{\pi}{n}, n = 1, 2, 3, \ldots, r = \theta\right\},$$

where (r, θ) are polar coordinates.

For each positive integer i, we consider a copy M_i of M with the diameter of M_i less than $1/i$ and we identify the point $(0, 0)$ of M_i with the point v_i of E. Let

$$Y = X \cup \bigcup_{i=1}^{\infty} M_i$$

with the identifications indicated above and the identification topology. That X is regular is well known, and it is then clear that Y is regular. Y is not strongly regular, since any finite set $S_n \subset Y$ such that each component of $Y \setminus S_n$ has diameter less than $1/n$ must contain points of E for sufficiently large n. Hence, for such an n, $Y \setminus S_n$ has infinitely many components, so that Y is not strongly regular.

The following example is due to A. Lelek.

EXAMPLE 2. *There is a curve X which is an inverse limit of connected graphs with monotone simplicial bonding maps and which is not the weakly confluent image of a dendrite.*

Proof. The curve X is constructed in the plane. For each positive integer n, let $a_n = (0, 1/n)$ and $b_n = (1/n, 0)$. Let $a_0 = (0, 2)$ and $b_0 = (2, 0)$. The curve X is the union of the straight line segment joining a_0 and $(0, 0)$, the straight line segment joining $(0, 0)$ and b_0, and the segments joining a_n and b_n. We shall show that there is no dendrite which can be mapped by a weakly confluent mapping onto X.

Suppose that D is a dendrite and that f is a weakly confluent mapping from D onto X. For each positive integer n let

$$A_n = \overline{a_0 a_n} \cup \overline{a_n b_n} \cup \overline{b_n b_0},$$

the union of three straight line segments. For each positive integer n there is a continuum C_n in D such that $f(C_n) = A_n$, since f is weakly confluent. For each positive integer n, f/C_n is weakly confluent, since any mapping from a continuum onto an arc is weakly confluent (see [9], p. 25). Therefore, for each positive integer n, there is a continuum K_n in C_n such that $f(K_n) = \overline{a_0 a_n}$. Similarly, there is a continuum L_n in C_n

such that $f(L_n) = \overline{b_0 b_n}$. The continua $\{K_n\}_{n=1}^{\infty}$ and $\{L_n\}_{n=1}^{\infty}$ have large diameter, so, since X is regular, there are infinitely many of the continua $\{K_n\}_{n=1}^{\infty}$ which have a point in common, say, $x \in \bigcap K_{n_i}$. Similarly, there is a point $y \in \bigcap L_{n_{i_j}}$ for infinitely many of the continua L_{n_i}. Hence, there are continua C_{n_0} and C_{n_1} $(n_0 < n_1)$ such that

$$\{x, y\} \subset C_{n_0} \cap C_{n_1} \subset f^{-1}(A_{n_0} \cap A'_{n_1}) = f^{-1}(\overline{a_0 a_{n_0}} \cup \overline{b_0 b_{n_0}}).$$

But $C_{n_0} \cup C_{n_1}$ is a subcurve of a dendrite, so $C_{n_0} \cup C_{n_1}$ cannot contain a simple closed curve. This is a contradiction.

We distinguish the following six classes of curves:

(I) regular curves;

(II) strongly regular curves;

(III) class $\mathscr{A}$;

(IV) inverse limits of connected graphs with monotone simplicial bonding maps;

(V) inverse limits of connected graphs with monotone simplicial retractions as bonding maps;

(VI) weakly confluent images of dendrites.

We shall show in Theorem 4 that (V) $\subset$ (VI). Tymchatyn has shown in [11] that (III) $\subset$ (I). We have observed that (II) $\subsetneq$ (I), and it is trivial that (V) $\subset$ (IV). A. Lelek has shown that (IV) $\subset$ (II). In fact, the inverse limits of strongly regular curves with monotone bonding maps are strongly regular. A. Lelek has also shown that (VI) $\subset$ (I).

Remark (see [7], Theorem 3.6). As a matter of fact, the class of regular curves is invariant under weakly confluent continuous transformations. To prove it, one can utilize a characterization of regular curves by means of properties of collections of disjoint subcontinua (see [8], p. 132). Since (IV) $\subsetneq$ (VI), it will follow from Theorem 4 that (IV) $\neq$ (V).

We include the following questions [1]:

(1) Is it true that each weakly confluent image of a dendrite belongs to class (IV)?

(2) Which strongly regular curves belong to (V)? (**P 991**)

(3) Which regular curves are weakly confluent images of dendrites?

(4) Is it true that (V) $\subset$ (III)?

(5) Is it true that (V) = (VI)?

(6) Is it true that (IV) = (II)?

(7) Is it true that (III) $\subset$ (II)?

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[1] E. D. Tymchatyn has recently answered questions (1), (4), and (5) in the affirmative, thereby answering question (3), and has provided a reply in the negative to questions (6) and (7) (see E. D. Tymchatyn, *Weakly confluent mappings and a classification of continua*, this fascicle, p. 229-233).

We wish to express our gratitude to Professor A. Lelek for his simplified version of the proof of our Theorem 1 which follows.

THEOREM 1. *If G is a connected graph, then there are a finite tree T and a weakly confluent mapping from T onto G.*

Proof. Suppose that G is a graph, and let U be the universal covering space of G with projection map p. Considering G as a 1-complex, let $\{S_i\}_{i=1}^k$ be the collection of all simplicial subcontinua of G. Let f_i be a mapping from $[0, 1]$ onto S_i for $i = 1, \ldots, k$. Let F_i be a lift of f_i into U for $i = 1, \ldots, k$. Define T_1 to be $\bigcup_{i=1}^{k} F_i([0, 1])$, and let T_2 be a continuum in U which contains T_1. The continuum T_2 is a finite tree. We observe that the projection map p restricted to T_2 is weakly confluent with respect to all simplicial subcontinua of G, that is, if S_i is a simplicial subcontinuum of G, then some of $p^{-1}(S_i)$ contains $F_i([0, 1])$ and the restriction of p to $F_i([0, 1])$ is a mapping onto S_i.

Let $\{v_i\}_{i=1}^n$ be the collection of all 0-simplexes of G, and let r_i be the ramification of V_i in G for $i = 1, \ldots, n$. For an integer i, $1 \leqslant i \leqslant n$, let $\{x_{i,j}\}_{j=1}^{l_i}$ be $p^{-1}(V_i) \cap T_2$. For each integer j, $1 \leqslant j \leqslant l_i$, we take r_i disjoint copies of the unit interval $[0, 1]_{k,j,i}$, $k = 1, \ldots, r_i$, and identify the point 0 in each of these intervals with the point $x_{i,j}$ with the identification topology.

Define T to be the finite tree

$$T_2 \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{l_i} \bigcup_{k=1}^{r_i} [0, 1]_{k,j,i}$$

with the above identifications and the identification topology.

We extend the mapping p to a mapping f from T onto G as follows. If i is an integer, $1 \leqslant i \leqslant n$, and if j is an integer, $1 \leqslant j \leqslant l_i$, then each of the r_i arcs $[0, 1]_{k,j,i}$, $k = 1, \ldots, r_i$, which is attached to $x_{i,j}$ is to be mapped linearly onto only one of the r_i arcs of G with one endpoint V_i with the condition that $x_{i,j}$ is mapped to V_i.

We wish to show that the mapping f is weakly confluent. Let K be a continuum in G. If K is a subset of some 1-simplex σ of G with one 0-face V_i for some integer i, $1 \leqslant i \leqslant n$, then some arc $[0, 1]_{k,j,i}$ for some integer k, $1 \leqslant k \leqslant r_i$, and for some integer j, $1 \leqslant j \leqslant l_i$, is mapped linearly onto σ with its endpoints by f, whence there is some component of $f^{-1}(K)$ which is mapped onto K by f.

Suppose that K is contained in no 1-simplex of G. Let M be the largest subcomplex of G which is contained in K. If M is not connected, then M has at least two components, C_1 and C_2, and C_1 and C_2 are connected subcomplexes of G. Since K is connected, there is an arc α in K from some point in C_1 to some point in C_2. Let P be the last point on this arc which lies in C_1. Since C_1 is a complex, P is a vertex of C_1. However,

a must contain a 1-simplex σ having one 0-face P and the other 0-face not in C_1. Therefore, since M is the largest subcomplex in K, σ is in M along with the 0-faces of σ, so σ is in C_1 along with the 0-faces of σ. This contradicts the fact that P is the last point of a which lies in C_1. Hence, M is a connected subcomplex of G. Since the restriction of P to T_2 is weakly confluent with respect to all connected subcomplexes of G, there is some component C of $f^{-1}(M)$ for which $f(C) = M$. Let Z be the collection of 0-simplexes of M, and $Z \subset \{V_i\}_{i=1}^n$. Each component of $K \setminus M$ is an open set in K lying in some 1-simplex of G and having only one limit point in M which is an element of Z; if S is one such component and $V_i \in Z$ is a limit point of S, then there is some integer j, $1 \leqslant j \leqslant l_i$, for which $x_{i,j} \in C$ and there is some arc $[0, 1]_{k,j,i}$ which is mapped linearly onto the 1-simplex in G which contains S with the condition that $x_{i,j} \in C$ is mapped onto V_i. We see from this that C altogether with all such arcs $[0, 1]_{k,j,i}$ contains a component of $f^{-1}(K)$ which is mapped onto K by f.

Remark. We note that the mapping f constructed in Theorem 1 is both finite-to-one and simplicial.

THEOREM 2. *Suppose that each of G_1 and G_2 is a connected graph, $G_1 \subset G_2$, and that D_1 is a finite tree. Suppose that f_1 is a weakly confluent, finite-to-one mapping from D_1 onto G_1, and assume that r is a monotone retraction from G_2 onto G_1. Then there exist a finite tree D_2, a monotone retraction t from D_2 onto D_1, and a weakly confluent, finite-to-one mapping f_2 from D_2 onto G_2 such that $r \circ f_2 = f_1 \circ t$.*

Proof. Suppose that G_1, G_2, D_1, f_1, and r are given as in the hypothesis. We wish to construct the finite tree D_2 by, so to speak, appropriately attaching finite trees to D_1.

Let $\{V_i\}_{i=1}^n$ be the set of vertices of G_1. For each i, $1 \leqslant i \leqslant n$, let $\{\omega_{i,j}\}_{j=1}^{k_i}$ be the set $f_1^{-1}(V_i)$. We note that this set is, in fact, finite, since f_1 is finite-to-one. Let G^i be $r^{-1}(V_i)$. Since r is a monotone retraction, G^i is a connected graph. Let D^i be a finite tree and let Ψ_i be a finite-to-one weakly confluent mapping from D^i onto G^i. We wish to attach several copies of D^i to each of the points $\omega_{i,j}$ for $j = 1, 2, \ldots, k_i$, and $i = 1, 2, \ldots, n$. For an integer i, $1 \leqslant i \leqslant n$, let $\{d^{i,l}\}_{l=1}^{p_i}$ be the finite set $\Psi_i^{-1}(V_i)$. We attach to each of the points $\omega_{i,j}$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, k_i$, exactly p_i copies of D^i at the points $d^{i,l}$ of D^i, $l = 1, 2, \ldots, p_i$, with the identification topology. Consequently, for an integer i, $1 \leqslant i \leqslant n$, we have attached $k_i p_i$ copies of D^i to D_1. Let D_2 be the finite tree thus defined.

We define mappings f_2 and t. If $x \in D_2$, then either $x \in D_1$ or there is an integer i, $1 \leqslant i \leqslant n$, for which $x \in D^i$. If $x \in D_1$, define $f_2(x)$ to be $f_1(x)$, and if $x \in D^i$, define $f_2(x)$ to be $\Psi^i(x)$. If $x \in D_1$, define $t(x)$ to be x, and if $x \in D^i$ attached to D_1 at the point $\omega_{i,j}$, define $t(x)$ to be $\omega_{i,j}$. Note that the functions f_2 and t are both well defined and continuous and that

$f_1 \circ t = r \circ f_2$. It is clear that the mapping t is a monotone retraction. We see also from the construction and the definition of f_2 that f_2 is finite-to-one. We have then only to show that f_2 is weakly confluent.

Let K be a continuum in G_2. Let $K_1 = K \cap G_1$. Since f_1 is weakly confluent, there is a component C of $f_1^{-1}(K_1)$ which is mapped onto all of K_1 by f_1. Considering D_1 as a subset of D_2, let $\{y_m\}_{m=1}^{q}$ be the collection of all vertices y of D_2 which have the property that $y \in C$ and y is a vertex of some finite tree D^i for some $i = 1, 2, \ldots, n$. Note that if V_i is a vertex of K_1, then $r^{-1}(V_i) \cap K$ is a continuum M_i in K which intersects K_1 at only the point V_i, since r is a monotone retraction. Hence, it suffices to show that there is some finite tree, say D^i, which has been attached to, say, the point $y_m \in \{y_m\}_{m=1}^{q}$ which has the property that there is a component C of D^i mapped onto all of M^i by f_2 and containing the point y_m. To see that this is so, we merely recall the way in which finite trees have been attached to D_1 at y_m; there are an integer i and an integer j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k_i$, such that $y_m = \omega_{i,j}$. p_i copies of the finite tree D^i are attached to the point $\omega_{i,j}$ at the points $\Psi_i^{-1}(V_i) = \{d^{i,l}\}_{l=1}^{p_i}$ of D^i. Hence, such a finite tree exists, since the maps Ψ_i are weakly confluent. Thus, f_2 is weakly confluent.

The following theorem is due to Read (see [10]).

THEOREM 3. *Suppose that a metric space Y is an inverse limit of an inverse system (Y_a, P_a^β, A), where each of the spaces Y_a is a compactum and each P_a^β is surjective. Suppose that f is a mapping from a compactum X onto Y such that $P_a \circ f$ is weakly confluent for each $a \in A$. Then f is weakly confluent.*

Proof. Suppose that $Y = \lim(Y_a, P_a^\beta, A)$, where Y_a is a compactum for each $a \in A$, and the map P_a^β is surjective for each a, $\beta \in A$. Suppose that f is a mapping from a compactum X onto Y such that, for each $a \in A$, $P_a \circ f$ is weakly confluent. Let C be a continuum in Y. Since, for each $a \in A$, $C_a = P_a(C)$ is a continuum in Y_a and $P_a \circ f$ is weakly confluent, there exists a continuum $K_a \subset X$ such that $[P_a \circ f](K_a) = C_a$ for each $a \in A$. There exists a cofinal subset B of A such that $\{K_\beta\}_{\beta \in B}$ converges to a limit continuum in X (see [5], p. 45).

Since B is cofinal in A, there is a homeomorphism h from Y onto $Y' = \lim\{Y_\beta, P_\gamma'^\beta, B\}$ such that $P'_\beta \circ h(y) = P_\beta(y)$ for each $y \in Y$ and $\beta \in B$. $h \circ f$ is a map from X to Y' such that $P'_\beta \circ h \circ f$ from X onto Y_β is weakly confluent for each $\beta \in B$, and $P'_\beta \circ f(C) = P_\beta(C)$ for each $\beta \in B$. If we can show that $h \circ f(K) = h(C)$, then $f(K) = C$. Hence we may assume without loss of generality that K is the limit of $\{K_a\}_{a A}$.

Let $y \in C$. For each $a \in A$ let $x_a \in K_a$ be such that $P_a \circ f(x_a) = P_a(y)$. There is an $x \in K$ such that x is a cluster point of the net $\{x_a\}_{a \in A}$. Suppose, for the sake of a contradiction, that $f(x) \neq y$. There is then an $a \in A$ such that $P_a \circ f(x) \neq P_a(y)$. Hence, there exist disjoint open sets U and V in Y_a such that $P_a \circ f(x) \in U$ and $P_a(y) \in V$. Thus $f^{-1}[P_a^{-1}(U)]$ is an open subset of X containing x. Hence, there is a $\beta > a$ such that $x_\beta \in f^{-1}[P_a^{-1}(U)]$. Therefore, we have

$$P_a(y) \in V \quad \text{and} \quad P_a(y) = P_a^\beta \circ P_\beta \circ f(x_\beta) = P_a[f(x_\beta)] \in U,$$

which is a contradiction. Hence, $f(x) = y$, and so $C \subset f(K)$.

Let $x \in K$. There is a net $\{x_a\}_{a \in A}$ which converges to x and such that each $x_a \in K_a$. Thus, $P_a \circ f(x_a) \in C_a$ for $a \in A$. Let

$$z_a \in P_a^{-1}[P_a \circ f(x_a)] \cap C.$$

Since C is compact, there is a $z \in C$ such that z is a cluster point of the net $\{z_a\}_{a \in A}$.

Suppose that $f(x) \neq z$. There is, therefore, an $a_0 \in A$ such that $P_{a_0}(x) \neq P_{a_0} \circ f(x)$. Let U and V be disjoint neighborhoods of $P_{a_0} \circ f(x)$ and $P_{a_0}(z)$, respectively. Since $\{x_a\}_{a \in A}$ converges to x, there is a β_0 such that if $\beta > \beta_0$, then $x_\beta \in f^{-1} \circ P_a^{-1}(U)$. Since $\{z_a\}_{a \in A}$ clusters to z, there is a $\gamma > \beta_0$, $\gamma > a_0$, such that $z_\gamma \in P_a^{-1}(V)$. Therefore,

$$x_\gamma \in f^{-1} \circ P_a^{-1}(U), \qquad P_a(z_\gamma) \in V \quad \text{and} \quad P_a \circ f(x_\gamma) \in U.$$

But,

$$P_a(z_\gamma) = P_\gamma^a \circ P_\gamma(z_\gamma) = P_\gamma^a \circ P_\gamma\big(f(x_\gamma)\big) = P_a\big(f(x_\gamma)\big)$$

which contradicts the assumption that U and V have no point in common. Hence, $f(x) = z$.

Hence, $f(K) = C$, so f is weakly confluent.

THEOREM 4. *If X is an inverse limit of connected graphs with monotone retractions as bonding maps, then there are a dendrite D and a weakly confluent mapping f from D onto X.*

Proof. Suppose that $X = \varprojlim(G_n, P_n)$, where, for each integer n G_n is a connected graph, and P_n is a monotone retraction from G_{n+1} onto G_n. It follows from Theorems 1 and 2 that for each integer n there is a finite tree D_n, there is a monotone retraction q_n from D_{n+1} onto D_n, and there is a weakly confluent finite-to-one map f_n from D_n onto G_n such that $P_n \circ f_{n+1} = f_n \circ q_n$. Let $D = \varprojlim(D_n, q_n)$. Then D is non-empty (see [3], p. 429), and it follows from [3], p. 430, that there is a mapping f from D onto X and, if p^n and q^n are the projections of X and D onto G_n and D_n, respectively, then we have $p^n \circ f = f_n \circ q^n$. The projection q^n is monotone (see [1], p. 240) and is, therefore, weakly confluent (see [2], p. 214). Hence $f_n \circ q^n = p^n \circ f$ is weakly confluent, since the class of weakly confluent mappings is multiplicative (see [6], p. 6). Therefore, by Theorem 5, f is weakly confluent.

We have only to show that D is a dendrite. It follows from [4], p. 72, that D is 1-dimensional and, by [1], p. 236, D is a continuum. Also, by [1], p. 241, D is locally connected and from [12], p. 249, we know that D contains no simple closed curve. Therefore, D is a dendrite.

LEMMA. *Let R be a strongly regular curve and let $\{T'_i\}_{i=1}^{\infty}$ be a sequence of finite subsets of R such that if i is a positive integer, then $R\setminus T'_i$ has only finitely many components each of which has diameter less than $1/i$. There is a sequence $\{T_i\}_{i=1}^{\infty}$ of finite subsets of R such that if i is a positive integer, then $R\setminus T_i$ has only finitely many components each of which has diameter less than $1/i$, $T_i \subset T_{i+1}$, and such that if P is a point of T_i, then P is a limit point of at least two components of $R\setminus T_i$.*

Proof. Let R and $\{T'_i\}_{i=1}^{\infty}$ be given as in the hypothesis. For a positive integer i, let

$$T''_i = \bigcup_{j\leqslant i}^{n} T'_j.$$

Clearly, each component of $R\setminus T''_i$ has diameter less than $1/i$; $R\setminus T''_i = R\setminus T'_i$ has only finitely many components. Suppose that $R\setminus T''_j$ has only finitely many components for $1\leqslant j\leqslant i$. Let S denote a component of $R\setminus T''_{i-1}$. Let $Q = \{q_j\}_{j=1}^{k}$ be the set of points of T_{i-1} which are in $\bar{S}$. Let $P = \{P_j\}_{j=1}^{m}$ be the set of points in $S\cap T'_i$. Let $D = S\setminus P$.

We wish to show that $D\cup Q$ has only finitely many components. Let U be a component of $D\cup Q$, and suppose that U is not a component of $R\setminus P$. Since $D\cup Q$ is locally connected, U is open in $D\cup Q$ and $U \subsetneqq W$, where W is a component of $R\setminus P$. $W\setminus U$ and U are not mutually separated. Hence, there is a point z in $U\cap\overline{W\setminus U}$ or in $\bar{U}\cap(W\setminus U)$. If $z\in U$, then either $z\in Q$ or $z\in D$, but since $U\cap D$ is open, $z\in Q$. If $z\in \bar{U}\setminus U$, then $z\in Q$, since $\bar{U}\setminus U\subset P\cup Q$. Therefore, each component of $D\cup Q$ is a component of $R\setminus P$ or contains a point of Q, so there are only finitely many such components.

We wish to show that D has only finitely many components. Suppose that D has infinitely many components. $D\cup Q$ is connected. D is open and each component of D is open. Hence, some point $p_0\in P$ is a limit point of infinitely many components of D, say $\{K_\alpha\}_{\alpha=1}^{\infty}$. Since $D\cup Q$ has only finitely many components, there are a point $q_0\in Q$ and an infinite subsequence $\{K_{\alpha_\beta}\}$ of $\{K_\alpha\}_{\alpha=1}^{\infty}$ such that q_0 is a limit point of K_{α_β}, $\beta = 1, 2, \ldots$ For $\beta = 1, 2, \ldots$, p_0 and q_0 are limit points of the connected set K_{α_β}. Hence, each open set containing p_0 whose closure does not contain q_0 must have a point of K_{α_β}, $\beta = 1, 2, \ldots$, on its boundary, contradicting the fact that R is regular. Therefore, $D = S\setminus P$ has only finitely many components, whence $R\setminus T''_i$ has only finitely many components.

We define the required $\{T_i\}_{i=1}^{\infty}$ as follows. Let B_i be the set of all points X of T''_i for which X is a limit point of only one component of

$R \setminus T_i''$. Let $T_i = T_i'' \setminus B_i$. The components of $R \setminus T_i''$ are the components of $R \setminus T_i$, so each component of $R \setminus T_i$ has diameter less than $1/i$ and there are only finitely many such components, and each point of T_i is a limit point of at least two components of $R \setminus T_i$. If $p \in T_i$, then p is a limit point of two components of $R \setminus T_i$, so p is a limit point of $R \setminus T_i''$. Hence $p \in T_{i+1}$. Therefore, $T_i \subset T_{i+1}$.

THEOREM 5. *Every strongly regular curve R is a monotone image of a curve belonging to class* (IV).

Proof. Let R be a strongly regular curve, and suppose, according to the Lemma, that $\{T_i\}_{i=1}^{\infty}$ is a sequence of finite subsets of R with $T_1 \subset T_2 \subset \ldots$ such that $R \setminus T_i$ has only finitely many components each of which has diameter less than $1/i$ and such that each point of T_i is a limit point of more than one component of $R \setminus T_i$. Label the components of $R \setminus T_i$, $\{B_{ij}\}_{j=1}^{k_i}$, and the points of T_i, $\{P_{il}\}_{l=1}^{m_i}$. Let

$$A = \{(B_{ij}, B_{ik}, P_{il}) \colon j < k \text{ and } P_{il} \text{ is a limit point of } B_{ij} \text{ and } B_{ik}\}.$$

For each $a_{jkl}^i \in A$, let a_{jkl}^i be a copy of $[0,1]$. Let G_i be the connected graph with arcs $\{a_{jkl}^i \colon a_{jkl} \in A\}$, where the left-hand endpoints of a_{jkl}^i and $a_{jk'l'}^i$ are identified, and the right-hand endpoints of a_{jkl}^i and $a_{j'kl'}^i$ are identified, and the left-hand endpoint of a_{jkl}^i is identified with the right--hand endpoint of $a_{k'jl'}^i$.

Define $f_{i+1} \colon G_{i+1} \to G_i$ as follows: if $x \in a_{jkl}^{i+1}$, where $P_{i+1\,l} \in B_{im}$, then $f_{i+1}(x)$ is to be the left-hand endpoint of $a_{mk'l'}$ or the right-hand endpoint of $a_{k'ml'}$. If $x = r \in a_{jkl}^{i+1}$, where $P_{i+1\,l} = P_{il}$ and $B_{i+1\,j} \subset B_{im}$ and $B_{i+1\,k} \subset B_{in}$, then define

$$f_{i+1}(x) = \begin{cases} r \in a_{mnl}^i & \text{for } m < n, \\ 1 - r \in a_{nml}^i & \text{for } m > n. \end{cases}$$

Let $L = \varprojlim\{G_i, f_i\}$. Then L is a curve belonging to class (IV).

We distinguish two types of points of L: $x \in L$ is of type I if, for some i, $f_i(x)$ is in the interior of some arc $a_{m'k'l'}^{i+i}$ of G_{i+1} and $P_l^i(x) = P_{l'}^{i+i}(x)$, and $x \in L$ is of type II if it is not of type I. In the latter case $f_i(x)$ is an endpoint of an arc of G_i for each i and we denote by B_i^x the component of $R \setminus T_i$ corresponding to that endpoint.

Define $f \colon L \to R$ as follows:

$$f(x) = \begin{cases} P_l^i & \text{if } x \text{ is of type I and } f_i(x) = a_{mkl}^i, \\ \bigcap \overline{B_i^x} & \text{if } x \text{ is of type II.} \end{cases}$$

We must show that f is monotone, continuous, and surjective.

We show first that f is surjective. Since, if $r \in \bigcup_{i=1}^{\infty} T_i$, r is an image of a point $x \in L$ of type I, we must show only that if $r \in R \setminus \bigcup_{i=1}^{\infty} T_i$, then there is a point $x \in L$ such that $f(x) = r$. Let

$$r \in R \setminus \bigcup_{i=1}^{\infty} T_i.$$

For each i, let B_i^r denote the component of $R \setminus T_i$ containing r. Then

$$r = \bigcap_i \overline{B_i^r}.$$

Note that, for each i, $B_i^r = B_{ij_i}$ has some point P_{il_i} of T_i as a limit point and $P_{l_i}^i$ is also a limit point of some other component B_{im_i} of $R \setminus T_i$. Hence, $a^i_{m_i j_i l_i}$ or $a^i_{j_i m_i l_i}$ is an arc of G_i. Denote by y_i the endpoint of $a^i_{m_i j_i l_i}$ or of $a^i_{j_i m_i l_i}$ that corresponds to B_i^r. Then, $f_{i+1}(y_{i+1}) = y_i$, since $B_{i+1}^r \subset B_i^r$. Therefore, there is a point $y \in L$ such that $f_i(y) = y_i$ and $f(y) = r$.

We wish to show that f is monotone. Let

$$r \in R \setminus \bigcup_{i=1}^{\infty} T_i$$

and consider distinct points x and y of $f^{-1}(r)$. Then x and y are of type II, $f_i(x) \neq f_i(y)$ for some i, and $B_i^x = B_i^y$. But $r \in B_i^x \cap B_i^y$. Therefore, $f^{-1}(r)$ is one point. Let $r \in T_i$. For each $j > i$ let G_j' denote the connected subgraph of G_j consisting of those arcs of G_j which correspond to $r \in T_j$. Let $L_r = \underleftarrow{\lim}\{G_j', f_{j|G_j'}\}$; the set of positive integers $z \geqslant i$ may be considered as a subset of L. We wish to show that $L_r = f^{-1}(r)$. Note that $f(L_r) = r$. Suppose $x \in L \setminus L_r$. Then, for some $j > i$, $f_j(x) \notin G_i'$. Thus, x is of type I, and $f_j(x)$ is in the interior of some arc of G_j corresponding to $s \in T_j$, $s \neq r$, so $f(x) = s$ or is of type II, and $f(x)$ is the endpoint of some arc of G_j that corresponds to some component B or $R \setminus T_i$ that does not have r as a limit point, and $f(x) \in \bar{B}$. Thus, $f(x) \neq r$.

We have only to show that f is continuous. Note that B, the collection of all connected open sets of R each of whose boundary is a subset of some T_i, is a basis for R. Suppose that $U \in B$, and that

$$\overline{U} \setminus U = \{t_1, t_2, \ldots, t_n\} \subset T_i \qquad f^{-1}(t_i) = L_{t_i}$$

is closed, so $f^{-1}(\overline{U} \setminus U)$ is closed. Let $x \in f^{-1}(U)$. Suppose that x is of type I. For each sufficiently large i, let $a^i_{j_i k_i l_i}$ be the arc of G_i such that $f_i(x)$ is in the interior of $a^i_{j_i k_i l_i}$. Since f_{i+1}^{-1}(the interior of $a^i_{j_i k_i l_i}$) $= a^{i+1}_{j_{i+1} k_{i+1} l_{i+1}}$ and $f_{i+1} | a^{i+1\,0}_{j_{i+1} k_{i+1} l_{i+1}}$ is a homeomorphism onto $a^i_{j_i k_i l_i}$ which is open in G_i, we may conclude that $\{z \in L\colon f_i(z) \in a^{i\,0}_{j_i k_i l_i}$ for sufficiently large $i\}$ is an open set of L containing x and lying in $f^{-1}(U)$. Suppose that x is of type II. Let ε be the distance from $f(x)$ to the boundary of U. Let N be a natural number such that $1/N < \varepsilon/4$. Let V be the union of the components $B_1^N, B_2^N, \ldots, B_k^N$ of $R\setminus T_N$ that have $f(x)$ as a limit point. Then $\overline{V} \subset U$. Let F_N be the subset of G_N consisting of the vertices corresponding to the components B_i^N together with the interiors of the arcs that have one of these vertices as an endpoint. F_N is open, so $(f_N)^{-1}(F_N)$ is open and contains x. We wish to show that $(f_N)^{-1}(F_N) \subset f^{-1}(U)$. Let $z \in (f_N)^{-1}(F_N)$. If $f_N(z)$ belongs to the interior of one of the arcs of F_N, then $f(z)$ is $f(x)$ or $f(z) \in \overline{V}\setminus V$, and $z \in f^{-1}(U)$. If $f_N(z)$ is a vertex of F_N corresponding to, say, B_1^N, then, if z is of type I so that $f_j(z)$ is in the interior of some arc of G_j, $j > N$, then the point corresponding to that arc is in $B_1^N \subset V \subset U$; if z is of type II so that $f_j(z)$ is some vertex of G_j corresponding to the component B_j^z of $R\setminus T_i$, then $f(z) = \bigcap \overline{B_i^z}$, but we have $B_N^z = B_1^N$, so that $f(z) \in \overline{B_N^1} \subset \overline{V} \subset U$. Therefore, $(f_N)^{-1}(F_N) \subset f^{-1}(U)$ and $f^{-1}(U)$ is open.

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