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ON SOME GENERALIZATION OF THE NOTION OF ASYMMETRY OF FUNCTIONS

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Introduction. There exists a number of mathematical papers dealing with sets of asymmetry and with sets of approximative asymmetry of functions. In the definition of asymmetry an essential part is being attached to the structure of the set of points satisfying the inequality $|f(x)-a| < \varepsilon$ in the vicinity of the point at which the asymmetry is to take place. It is worth while to be noted that the structure of such a set is also an indicator of continuity or of approximative continuity of a function. An attempt to give a uniform description of these, in a way, akin notions seem therefore to be purposeful. The result of my investigations in this direction is the notion of φ-regularity introduced in § 1 of this paper. Namely, it turns out that one may define the points at which a function is, in a sense, non-regular (e.g. it is uncontinuous, or has an asymmetry in the sense of Young) as points correspondingly non-regular with respect to some sets. This enables us to draw conclusions as to the magnitude of the set (in the sense of power, measure, category etc.) of points at which the behaviour of a function differs from the one which in the given case has been accepted as regular. Since I have mainly been concerned with asymmetry and its direct generalizations, the examples of non-regularity given in the paper refer to asymmetry. Continuity is considered only to show that the φ-regularity in question is a generalization of continuity.

1. φ-regularity of sets and functions.

Definition 1. Let X be a space and $\mathfrak{M}$ a σ-additive structure of subsets of this space. Moreover, let φ be an operation assigning a set $\varphi(A) \subset X$ to every set $A \in \mathfrak{M}$. (The complement of the set $\varphi(A)$ to the space X shall be denoted by $\varphi^*(A)$). The points of the set $\varphi(A)$ will be called *φ-regular* with respect to the set A, or *φ-regularity points* of the set A. Correspondingly points of the set $\varphi^*(A)$ will be called the *φ-non-regularity points* of the set A.

Definition 2. Let now Y be a topological separable space satisfying Hausdorf axiom and let f be a function defined in X whose values belong to the space Y. Further, let f be $\mathfrak{M}$-measurable, i.e. such that $f^{-1}(G) \in \mathfrak{M}$ for any open set G contained in the space Y. The function f is said to be *φ-regular* at the point $x \in X$ (or x is its *φ-regularity point*) if for every point $y \in Y$ and every neighbourhood U of it there exists a neighbourhood U' of this point such that $U' \subset U$ and $x \in \varphi(f^{-1}(U'))$.

The set of φ-regularity points of a function f shall be denoted by $\varphi(f)$. The points of the set $\varphi^*(f) = X - \varphi(f)$ shall be called φ-non-regularity points of the function f.

A fundamental theorem concerning the notions just introduced is the following

THEOREM 1. *Let a σ-additive structure $\mathfrak{M}$ and a σ-additive ideal $\mathfrak{N}$ of subsets of a space X be given. Let Y be a topological separable space containing at least two points.*

Then in order that for any $\mathfrak{M}$-measurable function f mapping X into Y we have $\varphi^*(f) \in \mathfrak{N}$ it suffices, and if $\mathfrak{M}$ is a σ-field it is also necessary, that for every $A \in \mathfrak{M}$ we have $\varphi^*(A) \in \mathfrak{N}$.

Proof. First, let $\varphi^*(f) \in \mathfrak{N}$ for every $\mathfrak{M}$-measurable function f. Let $A \in \mathfrak{M}$. If $\emptyset \neq A \neq X$, then by f we denote a two-valued function with values in Y such that $f(x_1) \neq f(x_2)$ when $x_1 \in A$ and $x_2 \notin A$. If, on the other hand, $A = X$ or A is an empty set, we take for f an arbitrary constant function with the value in Y. In both cases we obtain an $\mathfrak{M}$-measurable function. This follows, on the one hand, from the measurability of the set $X - A$ ($\mathfrak{M}$ is a field) and, on the other hand, from the fact that the points $y_1 = f(x_1)$ and $y_2 = f(x_2)$ $(x_1 \in A, x_2 \in X - A)$ have disjoint neighborhoods. We will show that

$$\varphi^*(A) \subset \varphi^*(f). \tag{1}$$

For, let first $A = X$ and y be the unique value of the function f. Then for every neighbourhood U' of the point y we have $f^{-1}(U') = X = A$. Thus $\varphi(f^{-1}(U')) = \varphi(A)$. Hence if now $x \in \varphi(f)$, then $x \in \varphi(A)$, which proves inclusion (1) in this case.

Now let A be an empty set. Consider a point $y \in Y$ which is not a value of the function f, and its neighbourhood U which does not contain any value of the function f. Thus the set $f^{-1}(U)$ is empty. Suppose that $x \in \varphi(f)$. Then there exists a neighbourhood $U' \subset U$ of the point y such that $x \in \varphi(f^{-1}(U'))$. Since the set $f^{-1}(U')$ is empty, and thus identical with A, we have $\varphi(f^{-1}(U')) = \varphi(A)$. Thus we have proved that $\varphi(f) \subset \varphi(A)$, and consequently inclusion (1) holds also in this case.

Finally, let $X \neq A \neq \emptyset$. Let y_1 and y_2 be the values taken by the function f in the set A and $X - A$ respectively. If U_1 is a neighbourhood of the point y_1 which does not contain the point y_2, then we have $f^{-1}(U_1) = A$, and thus inclusion (1) may be proved similarly as in the preceding case.

So we have proved inclusion (1). Hence the necessity of the condition asserted in Theorem 1 follows immediately.

Now we will prove the sufficiency of this condition. Suppose that $\varphi^*(A) \epsilon \mathfrak{N}$ for every set A belonging to the structure $\mathfrak{M}$. Let, further, $x \epsilon \varphi^*(f)$, where f stands for an $\mathfrak{M}$-measurable function. This means that there exists a point $y \epsilon Y$ and its neighbourhood U such that for every neighbourhood $U' \subset U$ of the point y we have $x \epsilon \varphi^*(f^{-1}(U'))$. In particular, we can take for U' a neighbourhood $V_n \subset U$ of the point y belonging to the basis of the space X.

Thus

$$\varphi^*(f) \subset \bigcup_{n=1}^{\infty} \varphi^*\big(f^{-1}(V_n)\big), \tag{2}$$

where $\{V_n\}$ denotes the sequence of all sets making up the basis of the space X. Since each of the sets $\varphi^*\big(f^{-1}(V_n)\big)$ belongs to the ideal $\mathfrak{N}$, the set $\varphi^*(f)$ is also an element of this ideal. Thus Theorem 1 has been proved.

Remark 1. The proof of the necessity of the condition given in Theorem 1 indicates that in this part of the theorem there is no need of assuming the separability of the space Y. Similarly, the sufficiency of this condition is ensured also when the space Y consists of only one point.

Remark 2. In the case when the space Y is finite, we may assume in Theorem 1 that $\mathfrak{N}$ is a finitely additive ideal.

2. Examples of φ-regularity. T-symmetry. In this paragraph, as well as in the next, we will give a number of examples of φ-regularity. These have been chosen to show the extension of the research field covered by this notion. This implies, unfortunately, the necessity of proving some known or even classical theorems as corollaries to Theorem 1. Only in Section 4 new results will be given.

I. Let X and Y be topological spaces, the space Y being separable. Put

$$\varphi(A) = (X-A) \cup \operatorname{Int}(A) \quad \text{for} \quad A \subset X. \tag{3}$$

It is not very difficult to see that the φ-regularity of the function mapping X into Y, given by formula (3), is equivalent to its continuity. Let now $\mathfrak{M}$ denote the family of open sets in the space X. The sets

$$\varphi^*(A) = A \cap \operatorname{Fr}(A) \tag{4}$$

are empty for $A \epsilon \mathfrak{M}$. Take now for $\mathfrak{N}$ a family consisting of an empty set.

The following is an immediate consequence of Theorem 1. If for every open set B in the space Y, the set $f^{-1}(B)$ is open in the space X, then the function f is continuous at every point of the space X.

Now, if $\mathfrak{M}$ is the family of F_σ-sets, then sets (4) are of the first category for $A \in \mathfrak{M}$. Thus taking for $\mathfrak{N}$ a family of the sets of the first category in X we obtain, as previously, a known theorem on the category of a set of discontinuity points of the function of the first class of Baire.

II. Let Y be the space of real numbers with the natural topology. Let X be a space whose elements are points of a euclidean n-dimensional space. In the space X we shall introduce a topology according to the following rule: A set $U \subset X$ is a neighbourhood of a point p belonging to U if the lower internal density of the set U at the point p is equal to unity. Let, in its turn, φ denote a function defined by formula (3) in the family of subsets of the space X. It can be easily verified that the φ--regularity of a function at a certain point means here the approximative continuity in the sense given by Denjoy. Take for $\mathfrak{M}$ a family of sets measurable according to Lebesgue. Then, for $A \in \mathfrak{M}$, $\varphi^*(A)$ is a set of measure zero. Thus denoting by $\mathfrak{N}$ the family of sets of measure zero we obtain the known theorem on the approximative continuity almost everywhere of a measurable function.

III. Let X be a euclidean space. Let in this space, besides the natural topology, a topology T be distiguished. In the sequel we will denote by $\bar{A}_T$, A'_T, etc. the closure of the set A, the set of its accumulations points etc. with respect to topology T. When the topology T coincides with the natural topology we will omit this additional notation.

Definition 3. A point $x \in X$ will be called a *T-asymmetry point* of the set $A \subset X$ if there exists an $(n-1)$-dimensional hyperplane H passing through x such that

$$x \in (A \cap X^+)'_T \Delta (A \cap X^-)'_T, \tag{5}$$

where X^+ and X^- denote components of the set $X-H$, while $A \Delta B$ denotes the symmetrical difference of the sets A and B. (In the sequel we will use the expression "There exists a decomposition $X^+ \cup H \cup X^-$ corresponding to the point x" instead of saying "there exists a hyperplane $H\ldots$".)

The points of the space X which are not T-asymmetry points of the set A shall be called its *T-symmetry points*.

By φ_T we shall denote an operation assigning to every set $A \subset X$ the set of its T-symmetry points. The φ_T-regularity of a function shall also be called its *T-symmetry*. In a similar way we introduce the notion of the asymmetry of functions.

Theorem 1, when applied to the notion of T-symmetry, takes the form of the following

THEOREM 2. *Let a σ-additive ideal $\mathfrak{N}$ of subsets of a space X be given. A necessary and sufficient condition in order that the set of T-asymmetry points of an arbitrary function mapping X into a separable space X (containing at least two elements) belongs to $\mathfrak{N}$ is that for every set $A \subset X$ the set of its T-asymmetry points belongs to $\mathfrak{N}$.*

Since the notions of asymmetry and approximative asymmetry have been introduced with the aid of limit values, upper and lower approximative limits and the like, it seems purposeful to introduce analogous notion in this case.

Definition 4. Let a function f map a set $A \subset X$ into the space Y. A point $y \in Y$ shall be called the *T-limit value* of the function f at point $x \in X$ if for any neighbourhood U of the point y

$$x \in \{f^{-1}(U')\}'_T. \tag{6}$$

We denote by $W_T(x, f, A)$ the set of T-limit values of the function $f|A$ at the point x. If T is a natural topology we will write $W(x, f, A)$ instead of $W_T(x, f, A)$.

The following theorem explains the relationship between the T-asymmetry and the structure of sets of T-limit values.

THEOREM 3. *A point $x \in X$ is a T-asymmetry point of a function f if and only if for every decomposition $X^+ \cup H \cup X^-$ of the space X corresponding to the point x we have*

$$W_T(x, f, X^+) = W_T(x, f, X^-). \tag{7}$$

We omit the easy proof of this theorem.

In particular, if X is a space consisting of real numbers ($n = 1$) we adopt the following notation: $W_T^+(x, f) = W_T\big(x, f, (x, +\infty)\big)$, $W_T^-(x, f) = W_T\big(x, f, (-\infty, x)\big)$. The points of the sets $W_T^+(x, f)$ and $W_T^-(x, f)$ will be called the *right-hand side* and *left-hand side T-limit values* of the function f at the point x, respectively. Thus a point x is a T-symmetry point of a function of one real variable if the set of its left-hand side T-limit values is identical with the set of its right-hand side T-limit values at the point x. (If we consider the natural topology in X, then we have to deal with the limit values in the usual sense.) In the next two paragraphs we shall be dealing with examples of T-asymmetry with various topologies. Paragraph 3 will be devoted to examples concerning functions of one variable, the power of the set of T-asymmetry points being our main subject of interest. In paragraph 4 a theorem on measure and category of the set of approximative asymmetry points of function of many variables will be given.

3. Examples of T-asymmetry on the straight line. Throughout this paragraph we denote by Y the set of real numbers with the usual topology. It is true that all results obtained in this paragraph may be carried over to the case when Y stands for a separable space containing at least two points. But because the aim of this paper is to deduce some theorems concerning real functions from Theorem 2, all the results of this paragraph shall be stated for the case mentioned above. Similarly, we denote by X the set of real numbers throughout this paragraph. In particular cases we apply to X various topologies obtaining thus various T-symmetries.

I. Let T denote the natural topology of the space X. It is easily verified that with this topology a point $x \in X$ is a T-asymmetry point of a set $A \subset X$ if and only if it is an accumulation point exactly of one of the sets $A \cap (x, +\infty)$ and $A \cap (-\infty, x)$. In such a case we shall say that x is a point of exactly *one-side accumulation* of the set A. The set $\varphi_T^*(A)$ is for every $A \subset X$ at most countable. To prove this it suffices to assign to every point $x \in \varphi_T^*(A)$ the adjoining interval which is a component of the complement of the set A. Thus we obtain a mapping of the set $\varphi_T^*(A)$ into a family of pairwise disjoint intervals, each of which corresponds to one or two points of the set $\varphi_T^*(A)$. Choosing now in Theorem 2 for $\mathfrak{N}$ a family of at most countable sets, we obtain a known theorem of Young [7] on the set of asymmetry points of a real function of one real variable.

II. Now we will show a comparatively large class of topologies in the space X which also allow to obtain a countable set of T-asymmetry points. Namely, the following theorem is true:

THEOREM 3. *Let the topology T be stronger than the usual topology in X and let it moreover satisfy the condition*

(W) *If $x_n \in (E_n)'_T$ for $n = 1, 2, \ldots$ and $x_n - x \to 0$ for $n \to \infty$, then $x \in (\bigcup E_n)'_T$.*

Then the set of T-asymmetry points of an arbitrary set $A \subset X$ and, consequently by Theorem 2, of an arbitrary real function of one real variable, is at most countable.

Proof. Let $A \subset X$ and $x \in \varphi_T^*(A)$. This means that x is a T-accumulation point of exactly one of the sets $A \cap (x, +\infty)$ and $A \cap (-\infty, x)$. We may assume that e.g. $x \in \{A \cap (x, +\infty)\}'_T$ and $x \notin \{A \cap (-\infty, x)\}'_T$. In view of condition (W), there exists a number $\delta > 0$ such that the interval $(x - \delta, x)$ does not contain any T-accumulation point of the set A. Thus this interval does not contain any T-asymmetry point of the set A, for the latter are its T-accumulation points. So we can establish, analogously as in the previous section, a mapping between the points of the set $\varphi_T^*(A)$ and a family of pairwise disjoint intervals. Thus we have proved that the set $\varphi_T^*(A)$ is countable.

Remark. Under the assumption that the topology T is stronger than the natural topology, condition (W) is equivalent to the following condition:

(W′) *For an arbitrary point $x \in X$ and its T-neighbourhood U there exists a number $\delta > 0$ such that the set $\{(x-\delta, x+\delta)-U\}'_T$ is empty.*

For let first the topology T satisfy condition (W′) and $x_n - x \to 0$ for $n \to \infty$ and $x_n \in (E_n)'_T$ for $n = 1, 2, \ldots$ Suppose that $x \notin (\bigcup E_n)'_T$. This means that there exists a T-neighbourhood U of the point x such that the set $U \cap (\bigcup E_n)$ does not contain any point different from x. By (W′) there exists a number $\delta > 0$ such that the set $(x-\delta, x+\delta)-U$ possesses no T-accumulation point. The convergence $x_n - x \to 0$, in its turn, implies the existence of a positive integer N such that $x-\delta < x_N < x+\delta$. Because the topology T is stronger than the natural topology, the interval $(x-\delta, x+\delta)$ is a T-neighbourhood of the point x_N. Thus x_N is a T-accumulation point of the set $(x-\delta, x+\delta) \cap E_N$, and, because of the choice of the number δ, it is a T-accumulation point of the set $E_N \cap U$. But this is impossible, for the set $E_N \cup U$ contains at most the point x. The contradiction obtained proves that $x \in (\bigcup E_n)'_T$, and thus condition (W′) implies condition (W).

Now let condition (W) be satisfied. Suppose there exists a point x and its neighbourhood U such that each of the sets $(x-\delta, x+\delta)-U$ possesses a T-accumulation point. In particular, each of the sets $(x-1/n, x+1/n)-U$ possesses a T-accumulation point x_n. Since the topology T is stronger than the natural topology, x_n is also an accumulation point of the set $(x-1/n, x+1/n)-U$ in the ordinary sense and thus the distance of the points x_n and x does not exceed $1/n$. Thus $x_n - x \to 0$. So, by condition (W), x is a T-accumulation point of the union of the sets $(x-1/n, x+1/n)-U$, and thus it is a T-accumulation point of each of them. This implies that x is a T-accumulation point of the complement of the set U which contradicts the fact that U is its T-neighbourhood. Thus condition (W) implies condition (W′) and their equivalence is proved.

Remark. In order that a topology stronger than the natural topology ensures the countability of the set of T-asymmetry points condition (W) is not necessary. To prove this let us consider the following example.

Let a topology T in which open sets are open intervals and a perfect nowhere dense set C (and all those sets which can be obtained from the latter by finite multiplications and arbitrary summations) be given. It follows from the definition of the topology T that it is stronger than the natural. But condition (W) is not satisfied. Indeed, let $\{E_n\}$denote a sequence of components of the set $X-C$ such that the sequence of their centres $\{x_n\}$ converges to a point $x_0 \in C$. Obviously we have $x_n \in (E_n)'_T$ but x_0 is not a T-accumulation point of the set $\bigcup E_n$. For the set C, as a T-neighbourhood of the point x_0, does not contain any point of the set $\bigcup E_n$.

On the other hand, sets of the form $\varphi_T^*(A)$ are countable. In fact, consider an arbitrary set $A \subset X$. Divide the points of the set $\varphi_T^*(A)$ into three classes. Let those points of the set $\varphi_T^*(A)$ which are the ends of the intervals adjoining to the set C belong to A_1. Let the set A_2 consist of all those T-asymmetry points of the set A which do not belong to C. The remaining part of the set $\varphi_T^*(A)$ is denoted by A_3. The countability of the set A_1 is obvious. The set A_2 consists of points of ordinary asymmetry of the set A and thus is countable. The points of the set A_3 are one-side accumulation points (in the ordinary sense) of the set $C \cap A$. Thus the set A_3 is countable and consequently so is the set $\varphi_T^*(A)$.

III. Let T_1 and T_2 be topologies of the set X defined as follows:

(a) U is said to be a T_1-*neighbourhood* of a point x if $x \in U$ and, moreover, if x is a density point of a measurable set $U' \subset U$.

(b) U is said to be a T_2-*neighbourhood* of a point x if it contains x and, moreover, if there exists a number $\delta > 0$ such that $(x-\delta, x+\delta)-U$ is a set of the first category.

Let $T = T_1 \cap T_2$. We will show that *in the topologies T and T_1 not all sets of the form $\varphi_T^*(A)$ are countable.*

Let A_2 be a set of the first category such that the measure of its complement is zero. Let A_1 consist of all the intervals of the form $(c-9^{-n}, c)$ such that the interval $(c-3^{-n}, c)$ is a component of the complement of a perfect Cantor set. Put $A = A_1 \cap A_2$. We will show that the sets $\varphi_T^*(A)$ and $\varphi_{T_1}^*(A)$ are uncountable. Consider namely the point

$$x = 2\sum_{k=1}^{\infty} 3^{-2^{n_k}}, \tag{8}$$

where $\{n_k\}$ stands for an arbitrary increasing sequence of positive integers. Put further

$$x_j = -9^{-2^{n_j}} + 2\sum_{k=1}^{j} 3^{-2^{n_k}}. \tag{9}$$

It is seen that $x_j < x$ and $\lim_{j\to\infty} x_j = x$. Moreover,

$$\frac{|A \cap (x_j, x)|}{x-x_j} \geqslant \frac{9^{-2^{n_j}}}{9^{-2^{n_j}} + 2\sum\limits_{k=j+1}^{\infty}} \geqslant \frac{4}{13}. \tag{10}$$

Thus the left-hand side upper density of the set A, as well as that of the set A_1 is positive. Hence x is a left-hand side T- and T_1-accumulation point of the sets A and A_1. On the other hand, the right-hand side density of each of those two sets at the point x is equal to zero. For if the interval $(x, x+\delta)$ and the set A_2 have a common segment (a, b) with the length d (this interval and the set A have a common part covering a segment of the same length up to a set of measure zero), then there exists in the same interval a segment adjoining the interval (a, b) from the left with the length $3^n d$ disjoint with the set A_1 (and consequently also with A). Here n increases indefinitely as δ decreases to 0. Thus x is a right-hand side T_1-accumulation point of neither of the sets A_1 and A. Hence x is a T_1-asymmetry point of the sets A and A_1. Since the set A is of the first category, it does not possess any T_2-accumulation points. Thus x is neither a right-hand side T-accumulation point of the set A. So we have proved that the points of the form (8) are the T-asymmetry points of the set A and the T_1-asymmetry points of the sets A_1 and A. Since there are as many points of the form (8) as there are increasing sequences of positive integers, the sets $\varphi_T^*(A)$, $\varphi_{T_1}^*(A)$ and $\varphi_{T_1}^*(A_1)$ are not countable. Since the T_1-asymmetry of a function is equivalent to the approximative asymmetry, the characteristic functions of the sets A and A_1 have uncountably many approximative asymmetry points.

Remark. The first example of a function having uncountably many approximative asymmetry points has been constructed by Belowska [1]. Her example is more complicated. The idea of introducing the characteristic function of a set related to Cantor set is due to Lipiński [5], who has also shown an example almost identical with the characteristic function of the set A.

IV. In Section III we considered simultanously several examples of T-asymmetry. An essential part in those considerations was played by the topology T_1 which is closely connected with the approximative asymmetry. The introducing of the topology T_2 is justified by the fact that with its aid we may develop some general considerations. Namely we will show that the countability and uncountability are not inherited (i.e. they are not hereditary properties with respect to the inclusion of topologies).

Namely let T_0 be the natural topology on the straight line and let T and T_2 have the same meaning as in Section 3. It is seen that $T_0 \subset T \subset T_2$. All sets of the form $\varphi_{T_2}^*(A)$ and $\varphi_{T_0}^*(A)$, where $A \subset X$ are countable (T_0 is the ordinary topology considered in Section I and T_2 satisfies condition (W)), which is not true for all sets of the form $\varphi_T^*(A)$.

4. Examples of T-symmetry in the n-dimensional space. Throughout this paragraph X stands for an n-dimensional euclidean space. Formulating the theorems on functions we shall confine ourselves to real functions, similarly as it was the case in the foregoing section.

I. Let T denote the natural topology of the space X. In this topology x is a T-asymmetry point of the set $A \subset X$ if there exists a decomposition $X^+ \cup H \cup X^-$ of the space X corresponding to the point x such that

$$x \in (A \cap X^+)' \Delta (A \cap X^-)'. \tag{11}$$

Similarly, x is a T-asymmetry point of a function f if there exists a decomposition $X^+ \cup H \cup X^-$ of the space X corresponding to the point x and such that

$$W(x, f, X^+) \neq W(x, f, X^-), \tag{12}$$

where $W(x, f, E)$ have the same meaning as in § 2. The T-asymmetry considered in this section shall be called briefly asymmetry.

It is easy to notice that if X is a plane and A its semiplane, then the asymmetry points of the set A fill the whole straight line. Thus the set of asymmetry points of a plane set must not be countable. So Young theorem on countability of the set of asymmetry points of a real function of one real variable cannot be carried over to functions of many variables. One may, however, prove the following

THEOREM 4. *The set of asymmetry points of a real function of n real variables is of the first category.*

Proof. Obviously, it suffices to prove that the sets $\varphi_T^*(A)$ are of the first category. We will even show that they are nowhere dense. Namely let $A \subset X$ and K be an arbitrary sphere in the space X. We will show that there exists a sphere $K_1 \subset K$ disjoint with the set $\varphi_T^*(A)$. First, if K does not contain any asymmetry points of the set A it suffices to put $K_1 = K$. So let $x \in K \cap \varphi_T^*(A)$. Consequently there exists a decomposition $X^+ \cup H \cup X^-$ corresponding to the point x and such that condition (11) is satisfied. Without loss of generality we may assume that $x \notin (A \cap X^+)'$. Thus there exists a sphere K_0 with its center in x such that the set $K_0 \cap X^+ \cap A$ is empty. Now let K_1 be a sphere contained in $K_0 \cap X^+$. The set $K_1 \cap A \subset K_0 \cap A \cap X^+$ is empty which implies that the sphere K_1 does not contain any accumulation point of the set A and consequently any of its asymmetry points. Thus theorem 4 is true.

II. Now let X, Y and T have the same meaning as in Section II of § 2. The T-symmetry corresponding to such topology will be called the *approximative symmetry*. Thus x is an approximative asymmetry point of a set $A \subset X$, if there exists a decomposition of the space X corresponding to the point x and such that one of the sets $A \cap X^+$ or $A \cap X^-$ has at the point x the external density equal to zero while the other has a positive external upper density.

The T-limit values of a function corresponding to the topology considered in this chapter will be called the *approximative limit values* of such a function. Accordingly, a point $y \in Y$ is an approximative limit value of the function f at the point $x \in X$ if for every neighbourhood U of the point y the set $f^{-1}(U)$ has at the point x a positive upper external density. Thus the approximative asymmetry of a function f at a point $x \in X$ denotes the existence of a decomposition $X^+ \cup H \cup X^-$ of the space X corresponding to the point x such that the sets of approximative limit values of the reduced functions $f|X^+$ and $f|X^-$ are not identical.

In the case when X is the set of real numbers the notions of approximative limit values and of approximative asymmetry introduced here are in full agreement with the analogous notions introduced by Kulbacka [4].

In the sequel, we will prove two theorems on the sets of approximative asymmetry points which generalize the corresponding results obtained by Kulbacka [4] for functions of one variable to functions of many variables.

THEOREM 5. *The approximative asymmetry points of a function of an arbitrary finite number of variables form a set of the first category.*

THEOREM 6. *The measure of a set of the approximative asymmetry points of a real function of a finite number of real variables is equal to zero.*

Before proving Theorems 5 and 6 we will introduce some notations useful in the proofs of this theorems.

Let X be a plane, $x \in X$, $\varepsilon > 0$ and let H be a straight line lying in X. By $P(x, \varepsilon, H)$ we denote a square situated in X with its center in x and with the sides 2ε such that the straight line H divides it in two rectangles.

By $A_{\beta,\delta}$, where $A \subset X$, $\beta > 0$ and $\delta > 0$, we denote the set of points x for which there exist decompositions $X^+ \cup H \cup X^-$ such that

$$(13) \qquad \frac{|A \cap P(x, \varepsilon, H) \cap X^-|}{|P(x, \varepsilon, H) \cap X^-|} = \frac{2|A \cap P(x, \varepsilon, H) \cap X^-|}{|P(x, \varepsilon, H)|} \leqslant \beta \quad \text{for} \quad 0 < \varepsilon < \delta$$

($|E|$ denotes the external measure of the set $E \subset X$).

We will show that the set $A_{\beta,\delta}$ is closed. In fact, let $x_n \in A_{\beta,\delta}$ for $n = 1, 2, \ldots$ and $x_n \to x$ for $n \to \infty$. Let further $X_n^+ \cup H_n \cup X_n^-$ denote the decompositions of the space X corresponding to the points x_n and satisfying condition (13) defining the set $A_{\beta,\delta}$. There exist a sequence $\{n_k\}$ of positive integers and a straight line H such that the angle between the straight lines H and H_{n_k} tends to zero as k indefinitely increases. Without reducing the generality of our considerations we may assume that $\{n_k\}$ is the sequence of all positive integers ordered according to their magnitude. Now let $p_n \in X_n^+$ and $q_n \in X_n^-$ be points which lie on one straight line perpendicular to H_n and whose distance from the point x_n is 1. As it can be easily proved the sequence $\{p_n\}$ contains a convergent subsequence. To avoid complicated notations let us assume that the sequence $\{p_n\}$ converges to a point p_0. Denote by X^+ that one of the components of the set $X-H$ to which belongs the point p_0, and the other by X^-. Not very complicated considerations show us that the decomposition $X^+ \cup H \cup X^-$ thus formed and the point x satisfy condition (13). This means that $x \in A_{\beta,\delta}$ and consequently that the set $A_{\beta,\delta}$ is closed.

We may now proceed to the proof of Theorems 5 and 6. To avoid cumbersome details in calculations, we shall prove both theorems for functions of two variables. In view of Theorem 2 it suffices to prove that for an arbitrary set $A \subset X$ the set $\varphi_T^*(A)$ is of the first category or of measure zero, respectively.

Proof of Theorem 5. Let $x \in \varphi_T^*(A)$. There exist numbers $\beta > 0$ and $\delta > 0$ such that $x \in A_{\beta,\delta}$ and the upper external density of the set A at the point x is not smaller than 10β.

Consider an arbitrary circle $K \subset X$ with the center in x and suppose that $K \subset A_{\beta,\delta}$. There exists a number $\varepsilon > 0$ such that $2\varepsilon < \delta$, $P(x, \varepsilon, H) \subset K$ and

$$\frac{|A \cap P(x, \varepsilon, H) \cap X^+|}{|P(x, \varepsilon, H) \cap X^+|} > 9\beta, \tag{14}$$

where $X^+ \cup H \cup X^-$ is a decomposition corresponding to the point x and satisfying condition (13). Denote by G_0 the rectangle $P(x, \varepsilon, H) \cap X^+$. Since it is included in $A_{\beta,\delta}$, there is a straight line H' passing through its center x' such that

$$\frac{|A \cap P(x', 2\varepsilon, H') \cap X'|}{|P(x', 2\varepsilon, H') \cap X'|} < \beta, \tag{15}$$

where X' stands for one of the semiplanes into which the straight line H' divides the plane X. Thus the straight line H' divides the rectangle G_0 into two congruent trapezes or triangles whose interiors are $X' \cap G_0$ and $X'' \cap G_0$ $\big(X'' = X-(X' \cup H')\big)$ in such a way that

$$|G_0 \cap X' \cap A| \leqslant 8\beta |G_0 \cap X'|. \tag{16}$$

Put $G_1 = X'' \cap G_0$.

Assume that the open sets $G_0, G_1, \ldots, G_k$ such that

$$G_{j-1} \supset G_j \quad \text{for} \quad 1 \leqslant j \leqslant k, \tag{17}$$

$$|G_j| = 2^{-j} |G_0| \tag{18}$$

and

$$\frac{|A \cap (G_0 - G_j)|}{|G_0 - G_j|} \leqslant 8\beta \quad \text{for} \quad 1 \leqslant j \leqslant k \tag{19}$$

have been defined. The set G_k may be represented as a union of rectangles with disjoint interiors. Applying to each of them an argument analogous to that carried out for the rectangle G_0, we can divide G_k into two subsets with equal measures in such a way that one of them, say G_{k+1}, be open, and on the other — the average external density of the set A — does not exceed β. Thus the existence of a sequence $\{G_j\}$ of subsets of the set G_0 such that relations (17), (18) and (19) are satisfied for any positive integer j has been proved. Hence

$$|A \cap P(x, \varepsilon, H) \cap X^+| = \big|A \cap \big(G_0 - (\cap G_j)\big)\big| \leqslant 8\beta |P(x, \varepsilon, H) \cap X^+|,$$

which contradicts (14). Thus the supposition that $K \subset A_{\beta,\delta}$ is not true. Consequently the set $A_{\beta,\delta}$ does not include any sphere with the center in x which means that x is a boundary point of the set $A_{\beta,\delta}$. Hence

$$\varphi_T^*(A) \subset \bigcup_{\beta,\delta} F_r(A_{\beta,\delta}), \tag{20}$$

where β and δ take any rational positive value. In view of the fact that the sets $A_{\beta,\delta}$ are closed, the sets $\mathrm{Fr}(A_{\beta,\delta})$ are nowhere dense and therefore the set $\varphi_T^*(A)$ is of the first category. Theorem 5 has thus been proved.

Proof of Theorem 6. Let $x \in \varphi_T^*(A)$. There exist, like in the foregoing proof, two rational numbers $\beta > 0$ and $\delta > 0$ such that $x \in A_{\beta,\delta}$ and the upper external density of the set at the point x is greater than 10β.

Suppose now that x is a density point of the set $A_{\beta,\delta}$. Hence, there exists a square K with the center in x whose sides are smaller than δ and such that $|K - A_{\beta,\delta}| \leqslant \beta|K|$ and $|K \cap A| > 10\beta|K|$. For every point $q \in K$ there exists an arbitrarily small square $K(q) \subset K$ with the center at q which may be divided into two congruent rectangles with the interiors $K^+(q)$ and $K^-(q)$ in such a way that $|K^-(q) \cap A| \leqslant \beta|K^-(q)|$. Thus the set $B_0 = K \cap A_{\beta,\delta}$ may be covered in the sense of Vitaly with squares D such that each of them is the union of two congruent trapezes (or triangles) [1] with the interiors D^+ and D^- such that $|A \cap D^-| \leqslant 8\beta|D^-|$. Thus there exists a sequence $\{D_n\}$ of pairwise disjoint rectangles covering the set B_0 except, may be, for a set of measure zero.

Put

$$B_1 = \bigcup_{n=1}^{\infty} (A_{\beta,\delta} \cap D_n^+)$$

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[1] The squares $K(q)$ need not have their sides parallel to the coordinate axes and Vitaly theorem on covering has been formulated in a particular case, therefore in our considerations we have to choose squares contained in squares $K(q)$ situated in accordance with the axes.

The example considered below shows that the existence of T-limit values depends on the space Y of values of a function. The existence of T-limit values is also influenced by points which are not values of the function. Let f have at a point x the left-hand side limit $-\infty$ and the right-hand side limit $+\infty$. If Y is the space of real numbers, then x is not an asymmetry point, for the sets $W^+(f, x)$ and $W^-(f, x)$ are empty and thus identical. But if we take for Y the set of real numbers with the points $+\infty$ and $-\infty$, then obviously, x is an asymmetry point.

REFERENCES

[1] L. Belowska, *Resolution d'un problème de M. Z. Zahorski sur les limites approximatives*, Fundamenta Mathematicae 48 (1960), p. 277-286.

[2] C. Goffman, *On the approximate limit of real functions*, Acta Scientiarum Mathematicarum, Szeged, 23 (1962), p. 76-78.

[3] E. Kamke, *Zur Definition der approximativ stetigen Funktionen*, Fundamenta Mathematicae 10 (1927), p. 431-433.

[4] M. Kulbacka, *Sur l'ensemble des points de l'asymétrie approximative*, Acta Scientiarum Mathematicarum, Szeged, 21 (1960), p. 90-95.

[5] J. S. Lipiński, *Sur la discontinuité approximative et la dérivée approximative*, Colloquium Mathematicum 10 (1963), p. 103-109.

[6] A. Matysiak, *Sur les limites approximatives*, Fundamenta Mathematicae 48 (1960), p. 363-366.

[7] W. H. Young, *La symétrie de structure des fonctions des variables réelles*, Bulletin des Sciences Mathématiques (2) 52 (1928), p. 265-280.

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