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ON SUBMANIFOLDS IN A SPACE WITH SASAKIAN 3-STRUCTURE

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0. Introduction. In differential geometry of submanifolds, it is important to study the relation between the submanifolds and the structure of ambient manifold.

For example, in Sasakian manifolds there are two typical classes of submanifolds. One is the class of invariant submanifolds and the other is the class of C-totally real (or anti-invariant) submanifolds. In an invariant submanifold, the Sasakian structure of the ambient manifold induces the same structure on the submanifold. In a C-totally real submanifold, a tangent vector of the submanifold is mapped into the normal space by the Sasakian structure φ.

There is a long history of the study of invariant submanifolds of the Sasakian manifold. The C-totally real submanifolds have been studied by Yamaguchi and Ikawa [11], Yamaguchi et al. [12], and Yano and Kon [14], [15].

On the other hand, Kuo [5] introduced the notion of almost contact 3-structure. Since then, Kashiwada [3], Kuo and Tachibana [6], Sasaki [7], Tachibana and Yu [9], and Tanno [10] studied several interesting subjects concerning this structure or the Sasakian 3-structure.

In this paper, we shall study the relation between the Sasakian 3-structure and submanifolds.

1. Submanifolds. First we recall the fundamental properties of submanifolds in a Riemannian manifold. Let $\bar{M}$ be an m-dimensional Riemannian manifold and let M be an n-dimensional submanifold isometrically immersed in $\bar{M}$. Let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M and $\mathfrak{X}(M)^{\perp}$ the set of all vector fields normal to M. We denote by $\langle\, ,\rangle$ the Riemannian metric tensor field on $\bar{M}$ as well as the metric induced on M. The operator of covariant differentiation of $\bar{M}$ (respectively, M) will be denoted by $\bar{\nabla}$ (respectively, ∇). Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X N = -A^N(X) + D_X N,$$
$$X, Y \in \mathfrak{X}(M), \; N \in \mathfrak{X}(M)^{\perp},$$

where D denotes the operator of covariant differentiation with respect to the linear connection induced on the normal bundle of M, and A, B are both called the second fundamental forms of M which are related by $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$.

The *second fundamental form* B is a vector bundle valued symmetric bilinear form on each tangent space $T_x(M)$ ($x \in M$) taking values in the normal space $T_x(M)^{\perp}$. The *second fundamental form* A is a cross-section of a vector bundle $\mathrm{Hom}(T(M)^{\perp}, S(M))$, where $S(M)$ denotes the bundle whose fiber at each point is the space of symmetric linear transformations $T_x(M) \to T_x(M)$; i.e., for $N \in T_x(M)^{\perp}$ we have

$$A^N : T_x(M) \to T_x(M).$$

We define a covariant derivative $(\tilde{\nabla}B)$ of the second fundamental form B by

$$(\tilde{\nabla}_X B)(Y, Z) = D_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),$$
$$X, Y, Z \in \mathfrak{X}(M).$$

If $\tilde{\nabla}_X B = 0$ for all $X \in \mathfrak{X}(M)$, then the second fundamental form B of M is said to be *parallel*. This is equivalent to $(\nabla_X A) = 0$, where $(\nabla_X A)$ is defined by

$$(\nabla_X A)^N(Y) = \nabla_X\big(A^N(Y)\big) - A^{D_X N}(Y) - A^N(\nabla_X Y),$$
$$X, Y \in \mathfrak{X}(M), N \in \mathfrak{X}(M)^{\perp}.$$

If the second fundamental form B is identically zero, then M is said to be *totally geodesic*. A normal vector field N on M is said to be *parallel* if $D_X N = 0$ for any tangent vector field X.

Let $\bar{R}$ and R be the Riemannian curvature tensor fields of $\bar{M}$ and M, respectively. Then we have

$$(1.1) \quad \bar{R}(X, Y)Z = R(X, Y)Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y) +$$
$$+ (\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z), \quad X, Y, Z \in \mathfrak{X}(M).$$

Hence we have the equation of Gauss:

$$(1.2) \quad \langle \bar{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle + \langle B(Y, W), B(X, Z)\rangle -$$
$$- \langle B(X, W), B(Y, Z)\rangle, \quad X, Y, Z, W \in \mathfrak{X}(M).$$

Taking the normal part of (1.1), we have the equation of Codazzi:

$$(\bar{R}(X, Y)Z)^{\perp} = (\tilde{V}_X B)(Y, Z) - (\tilde{V}_Y B)(X, Z).$$

We define the curvature tensor R of the normal bundle of M by

$$R^{\perp}(X, Y)N = [D_X, D_Y]N - D_{[X,Y]}N, \quad X, Y \in \mathfrak{X}(M), N \in \mathfrak{X}(M)^{\perp}.$$

Then we get the equation of Ricci:

$$\langle \bar{R}(X, Y)N_1, N_2 \rangle = \langle R^{\perp}(X, Y)N_1, N_2 \rangle + \langle [A^{N_1}, A^{N_2}](X), Y \rangle,$$
$$X, Y \in \mathfrak{X}(M), \; N_1, N_2 \in \mathfrak{X}(M)^{\perp}.$$

If $R^{\perp}$ vanishes identically, then the normal connection of M is said to be *flat*. If $[A^{N_1}, A^{N_2}] = 0$ for any $N_1, N_2 \in \mathfrak{X}(M)^{\perp}$, then the second fundamental form of M is said to be *commutative*.

2. Sasakian 3-structure. In this section we recall the definition and some properties of Sasakian 3-structure (see [2]). Let M be a Riemannian manifold with metric tensor $\langle , \rangle$. We denote by ∇ the covariant differentiation of M.

Assume that M admits three unit Killing vectors ξ, η, and ζ which are mutually orthogonal and satisfy

$$2\xi = [\eta, \zeta], \quad 2\eta = [\zeta, \xi], \quad 2\zeta = [\xi, \eta].$$

We put

$$\varphi = \nabla\xi, \quad \psi = \nabla\eta, \quad \theta = \nabla\xi \quad \text{and} \quad \Phi = \nabla\alpha, \quad \Psi = \nabla\beta, \quad \Theta = \nabla\gamma,$$

where α, β, and γ are 1-forms associated with ξ, η, and ζ, respectively; i.e., $\alpha(X) = \langle X, \xi \rangle$, $\beta(X) = \langle X, \eta \rangle$, and $\gamma(X) = \langle X, \zeta \rangle$. Since ξ, η, and ζ are Killing vectors, Φ, Ψ, and Θ are skew-symmetric $(0, 2)$-tensor fields and satisfy

$$\varphi\xi = 0, \quad \psi\eta = 0, \quad \theta\zeta = 0, \tag{2.1}$$

$$d\Phi = 0, \quad d\Psi = 0, \quad d\Theta = 0, \quad \nabla_\xi\varphi = 0, \quad \nabla_\eta\psi = 0, \quad \nabla_\zeta\theta = 0.$$

On the other hand, we have

$$\psi\zeta + \theta\eta = 0, \quad \theta\xi + \psi\zeta = 0, \quad \varphi\eta + \psi\xi = 0,$$

$$\xi = \theta\eta = -\psi\zeta, \quad \eta = \varphi\zeta = -\varphi\xi, \quad \zeta = \psi\xi = -\varphi\eta \tag{2.2}$$

because ξ, η and ζ are unit normal Killing vectors. From these equations we get

$$\nabla_\xi\xi = 0, \quad \nabla_\eta\eta = 0, \quad \nabla_\zeta\zeta = 0, \quad \nabla_\eta\zeta = -\nabla_\zeta\eta = \xi,$$
$$\nabla_\zeta\xi = -\nabla_\xi\zeta = \eta, \quad \nabla_\xi\eta = -\nabla_\eta\xi = \zeta.$$

A triple $\{\xi, \eta, \zeta\}$ of Killing vectors is called a ***K-contact 3-structure*** if all of ξ, η, and ζ are K-contact structures. That is, if $\{\xi, \eta, \zeta\}$ is a K-contact 3-structure, then it satisfies

$$\begin{aligned} &\varphi^2 = -I+\alpha\otimes\xi, \quad &\psi^2 = -I+\beta\otimes\eta, \quad &\theta^2 = -I+\gamma\otimes\zeta, \\ &\theta\psi = \varphi+\beta\otimes\zeta, \quad &\varphi\theta = \psi+\gamma\otimes\xi, \quad &\psi\varphi = \theta+\alpha\otimes\eta, \\ &\psi\theta = -\varphi+\gamma\otimes\eta, \quad &\theta\varphi = -\psi+\alpha\otimes\zeta, \quad &\varphi\psi = -\theta+\beta\otimes\xi, \end{aligned} \tag{2.3}$$

where I denotes the identity transformation.

A K-contact structure $\{\xi, \eta, \zeta\}$ is called a ***Sasakian 3-structure*** or a ***normal contact metric 3-structure*** if all of ξ, η, and ζ are Sasakian structures.

If $\{\xi, \eta, \zeta\}$ is a Sasakian 3-structure, then it satisfies

$$\begin{aligned} \nabla_X\varphi &= -X\otimes\xi+\alpha\otimes X = R(X, \xi), \\ \nabla_X\psi &= -X\otimes\eta+\beta\otimes X = R(X, \eta), \\ \nabla_X\theta &= -X\otimes\zeta+\gamma\otimes X = R(X, \zeta). \end{aligned}$$

If for a K-contact 3-structure $\{\xi, \eta, \zeta\}$ any two of ξ, η, and ζ are Sasakian structures, then $\{\xi, \eta, \zeta\}$ is necessarily a Sasakian 3-structure. If a Riemannian manifold M has a Sasakian 3-structure, then M is necessarily of dimension $4m+3$ ($m \geqslant 0$ is an integer) and orientable.

Let M be a Riemannian manifold with Sasakian 3-structure. Then the curvature tensor $R(X, Y)$ of M satisfies

$$\begin{aligned} \langle R(X, Y)Z, W\rangle - \langle R(X, Y)\varphi Z, \varphi W\rangle &= \langle X, W\rangle\langle Y, Z\rangle - \\ &-\langle X, Z\rangle\langle Y, W\rangle + \langle\varphi X, Z\rangle\langle Y, \varphi W\rangle - \langle\varphi Y, Z\rangle\langle X, \varphi W\rangle, \\ \langle R(X, Y)Z, W\rangle - \langle R(X, Y)\psi Z, \psi W\rangle &= \langle X, W\rangle\langle Y, Z\rangle - \\ &-\langle X, Z\rangle\langle Y, W\rangle + \langle\psi X, Z\rangle\langle Y, \psi W\rangle - \langle\psi Y, Z\rangle\langle X, \psi W\rangle, \\ \langle R(X, Y)Z, W\rangle - \langle R(X, Y)\theta Z, \theta W\rangle &= \langle X, W\rangle\langle Y, Z\rangle - \\ &-\langle X, Z\rangle\langle Y, W\rangle + \langle\theta X, Z\rangle\langle Y, \theta W\rangle - \langle\theta Y, Z\rangle\langle X, \theta W\rangle. \end{aligned}$$

The following propositions are well known.

PROPOSITION 2.1 ([2]). *In a Riemannian manifold with triple $\{\xi, \eta, \zeta\}$ of Killing vectors, any integral manifold of the distribution spanned by ξ, η, and ζ is totally geodesic and of constant curvature* 1.

PROPOSITION 2.2 ([3]). *A Riemannian manifold with Sasakian 3-structure is an Einstein space.*

We denote by D the subspace of $T_x(M)$ defined by

$$D = \{X \mid X \in T_x(M),\ \alpha(X) = \beta(X) = \gamma(X) = 0\}.$$

Let $K(X, Y)$ be the sectional curvature of M spanned by unit vectors X and Y. For a unit vector X, we put

$$H_\xi(X) = K(X, \varphi X), \quad H_\eta(X) = K(X, \psi X), \quad H_\zeta(X) = K(X, \theta X).$$

Then we have the following

PROPOSITION 2.3 ([10]). *For any unit tangent vector X ($X \in D$) we have* $H_\xi(X) + H_\eta(X) + H_\zeta(X) = 3$.

Typical examples of the manifold with Sasakian 3-structure are odd-dimensional spheres S^{4m+3}, projective spaces $S^{4m+3}/\{\pm 1\}$, and lens spaces (see [7]).

Let QP^m be the quaternionic projective space. A unit sphere $S^{4m+3}(1)$ and QP^m are related by the Hopf fibering, i.e.:

$$\pi\colon\ S^{4m+3}(1) \to S^{4m+3}(1)/(\xi, \eta, \zeta) = QP^m.$$

Let M be a Sasakian manifold with Sasakian structure $(\varphi, \alpha, \xi, \langle\ ,\ \rangle)$. A plane section in the tangent space $T_x(M)$ is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature $K(X, \varphi X)$ with respect to a φ-section determined by a vector X is called a *φ-sectional curvature*. It is easily verified that if a Sasakian manifold M has a φ-sectional curvature k which does not depend on the φ-section at each point, then k is a constant in the manifold. A Sasakian manifold is called a *Sasakian space form* and is denoted by $M(k)$ if it has the constant φ-sectional curvature k. The curvature tensor $R(X, Y)$ of a Sasakian space form $M(k)$ is given by

$$\begin{aligned} (2.4)\qquad 4R(X, Y)Z = {} & (k+3)[\langle Y, Z\rangle X - \langle X, Z\rangle Y] + \\ & + (k-1)[\alpha(X)\alpha(Z)Y - \alpha(Y)\alpha(Z)X + \\ & + \langle X, Z\rangle \alpha(Y)\xi - \langle Y, Z\rangle \alpha(X)\xi - \langle Y, \varphi Z\rangle \varphi X + \\ & + \langle X, \varphi Z\rangle \varphi Y + 2\langle X, \varphi Y\rangle \varphi Z]. \end{aligned}$$

3. Submanifolds of a space with Sasakian 3-structure. In this section we shall define and study some submanifolds of a Riemannian manifold with Sasakian 3-structure.

Let $\bar{M}$ ($\dim \bar{M} = 4m+3$) be a Riemannian manifold with Sasakian 3-structure $\{\xi, \eta, \zeta\}$. We denote by M a submanifold of $\bar{M}$.

Definitions. 1. We say that M is *invariant with respect to* ξ or that (ξ, φ) is *invariant on* M if ξ is tangent to M and φX is tangent to M for any tangent vector X of M.

2. We say that M is *anti-invariant with respect to* ξ or that (ξ, φ) is *anti-invariant on* M if ξ is tangent to M and φX is normal to M for any tangent vector X of M.

3. We say that M is *C-totally real with respect to* ξ or that (ξ, φ) is *C-totally real on* M if ξ is normal to M.

Clearly, we can give the same definitions for (η, ψ) and (ζ, θ).

If a submanifold M is C-totally real with respect to ξ and η and invariant with respect to ζ, then without loss of generality it is equivalent to say that M is C-totally real with respect to η and ζ and invariant with respect to ξ. Therefore, in this case we can simply say that M is a *2-C-totally real and 1-invariant submanifold.* For other terminology we shall use the same convenience.

If M is C-totally real with respect to ξ, then φX is normal to M for any tangent vector X of M (cf. [11]).

PROPOSITION 3.1. *There is no 3-anti-invariant submanifold in a Riemannian manifold with Sasakian 3-structure.*

Proof. Assume that (ξ, φ) and (η, ψ) are anti-invariant. By (2.2) we have $\zeta = \psi\xi$, i.e., ζ is normal to M. Hence θX is normal to M for any tangent vector X of M. On the other hand, $\theta\xi = -\eta$ is tangent to M by virtue of (2.3). This is a contradiction.

THEOREM 3.1. *Let M be a 3-invariant submanifold in a Riemannian manifold $\bar{M}$ with Sasakian 3-structure. Then M is totally geodesic.*

Proof. For the second fundamental form B we have (see [11])

$$B(X, \varphi Y) = \varphi B(X, Y), \quad B(X, \psi Y) = \psi B(X, Y),$$
$$B(X, \theta Y) = \theta B(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$.

By (1.2) we have

$$\langle \bar{R}(X, \varphi X)\varphi X, X\rangle = \langle R(X, \varphi X)\varphi X, X\rangle + \langle B(\varphi X, X), B(X, \varphi X)\rangle - \\ - \langle B(X,X), B(\varphi X, \varphi X)\rangle$$

for any tangent vector X of unit length.

Hence

$$\bar{H}_\eta(X) = H_\eta(X) + 2\|B(X, X)\|^2, \quad \bar{H}_\zeta(X) = H_\zeta(X) + 2\|B(X, X)\|^2,$$
$$\bar{H}_\xi(X) = H_\xi(X) + 2\|B(X, X)\|^2.$$

Consequently, we have

$$\bar{H}_\xi(X) + \bar{H}_\eta(X) + \bar{H}_\zeta(X) = H_\xi(X) + H_\eta(X) + H_\zeta(X) + 6\|B(X, X)\|^2.$$

Therefore, from this equation, by Proposition 2.3, we have $B(X, X) = 0$. Hence M is totally geodesic.

4. 2-totally real submanifolds. Let $\bar{M}$ be a $(4m+3)$-dimensional Riemannian manifold with Sasakian 3-structure. Let M $(\dim M = 2m+1)$ be a submanifold of $\bar{M}$. We assume that M is totally real with respect to (ξ, φ) and (η, ψ). Since $\zeta = \psi\xi$ and $\theta X = \psi\varphi X$, we can see that (ζ, θ) is invariant by virtue of the dimensional condition. For the second fundamental form of the submanifold M, we have

$$\begin{gathered}
A^{\xi}(X) = 0, \quad \varphi X = D_X \xi, \quad A^{\varphi X}(Y) = A^{\varphi Y}(X), \quad A^{\eta}(X) = 0, \\
\psi X = D_X \eta, \quad A^{\psi X}(Y) = A^{\psi Y}(X), \quad B(X, \zeta) = 0, \\
B(X, \theta Y) = \theta B(X, Y), \\
A^N \theta = -A^{\theta N} = -\theta A^N, \quad X, Y \in \mathfrak{X}(M), N \in \mathfrak{X}(M)^{\perp}.
\end{gathered} \tag{4.1}$$

From (4.1) it follows that M is a minimal submanifold of $\bar{M}$. Using calculations similar to those of [12], we have

PROPOSITION 4.1. *Let M be a $(2m+1)$-dimensional 2-totally real and 1-invariant submanifold of a Riemannian manifold $\bar{M}$ $(\dim \bar{M} = 4m+3)$ with Sasakian 3-structure. Then the submanifold M is of constant curvature 1 if and only if the normal connection is flat.*

PROPOSITION 4.2. *Under the same assumption as in Proposition 4.1, if the second fundamental form of M is parallel, then M is totally geodesic.*

Next we shall study the integral formulas of 2-totally real and 1-invariant submanifolds. Let M be a $(2m+1)$-dimensional 2-totally real and 1-invariant submanifold in a unit sphere $S^{4m+1}(1)$. Let N_i be any unit normal vector of M. Then, for simplification, we write A^i instead of A^{N_i}. We choose a local field of orthonormal frame $\{e_1, \ldots, e_m; \theta e_1, \ldots, \theta e_m; e_{2m+1} = \zeta\}$ in M. Then we can see that the normal space $T_x(M)^{\perp}$ is spanned by $\{\varphi e_1, \ldots, \varphi e_m; \psi e_1, \ldots, \psi e_m; \xi, \eta\}$. Unless otherwise stated, we use the conventions that the ranges of indices are the following:

$$I = 1, \ldots, 2m, 2m+1,$$

$$i, j = 1, \ldots, 2m, \quad a, b = 1, \ldots, 2m, 2m+1, 2m+2.$$

We remark that $A^{\xi}(X) = 0$, $A^{\eta}(X) = 0$, and $A(\zeta) = 0$.

Since M is minimal, we have the following Simons' type formula [7]:

$$\nabla^2 A = (2m+1)A - A \circ \tilde{A} - \underset{\sim}{A} \circ A, \tag{4.2}$$

where the operators $\tilde{A}$ and $\underset{\sim}{A}$ are defined by

$$\tilde{A} = {}^t A \circ A \quad \text{and} \quad \underset{\sim}{A} = \sum_a (\operatorname{ad} A^a) \operatorname{ad} A^a.$$

We get

$$\underset{\sim}{A} = \sum_i (\mathrm{ad}\, A^i)\mathrm{ad}\, A^i$$

by the remark stated above.

In the sequel we need the following lemma stated in [1]:

LEMMA 4.1. *Let A and B be symmetric (n, n)-matrices. Then*

$$-\mathrm{Tr}(AB-BA)^2 \leqslant 2\mathrm{Tr}\, A^2\, \mathrm{Tr}\, B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$, respectively, where

$$\bar{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \quad \bar{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Moreover, if A_1, A_2, and A_3 are three symmetric matrices such that

$$-\mathrm{Tr}(A_k A_l - A_l A_k)^2 = 2\mathrm{Tr}(A_k)^2\,\mathrm{Tr}(A_l)^2, \qquad 1 \leqslant k, l \leqslant 3,$$

then at least one of the matrices A_k must be zero.

Using (4.1) we have the following lemma (cf. [12]):

LEMMA 4.2. *Let M be a $(2m+1)$-dimensional 2-totally real and 1-invariant submanifold in a Riemannian manifold $\bar{M}$ $(\dim \bar{M} = 4m+3)$ with Sasakian 3-structure. Then the second fundamental form A of M satisfies $\langle \nabla A, \nabla A\rangle - 4\|A\|^2 \geqslant 0$.*

We put $A_{ij} = \mathrm{Tr}(A^i A^j)$ and $A_i = A_{ii}$. Since A_{ij} is a symmetric $(2m, 2m)$-matrix, we can assume it is diagonal for a suitable frame. We have

$$\sum_i A_i = \|A\|^2.$$

From (4.2) it follows that

$$\begin{aligned}\langle \nabla^2 A, A\rangle &= (2m+1)\|A\|^2 - \langle A\circ\tilde{A}, A\rangle - \langle \underset{\sim}{A}\circ A, A\rangle \\ &= (2m+1)\|A\|^2 - \sum_i (A_i)^2 + \sum_{i\neq j}\mathrm{Tr}(A^i A^j - A^j A^i)^2.\end{aligned}$$

Using Lemma 4.1, we have

(4.3)

$$-\langle \nabla^2 A, A\rangle = -(2m+1)\|A\|^2 + \sum_i (A_i)^2 - \sum_{i\neq j}\mathrm{Tr}(A^i A^j - A^j A^i)^2$$

$$\leqslant -(2m+1)\|A\|^2+2\sum_{i\neq j}A_iA_j+\sum_i (A_i)^2$$

$$=\left[\left(2-\frac{1}{2m}\right)\|A\|^2-(2m+1)\right]\|A\|^2-\frac{1}{2m}\sum_{i>j}(A_i-A_j)^2.$$

If M is compact, we have

$$0=\frac{1}{2}\int_M \Delta\|A\|^2=\int_M \{\langle \nabla A,\ \nabla A\rangle+\langle \nabla^2 A,\ A\rangle\}.$$

Lemma 4.2, (4.3), and this integral formula imply the following

THEOREM 4.1. *Let M be a $(2m+1)$-dimensional compact 2-totally real and 1-invariant submanifold in a unit sphere $S^{4m+3}(1)$. Then*

$$\int_M \left[\left\{\left(2-\frac{1}{2m}\right)\|A\|^2-2(m+5)\right\}\|A\|^2-\frac{1}{2m}\sum_{i>j}(A_i-A_j)^2\right]\geqslant 0.$$

THEOREM 4.2. *Let M be a $(2m+1)$-dimensional compact 2-totally real and 1-invariant submanifold in a unit sphere $S^{4m+3}(1)$. If the second fundamental form A satisfies*

$$\|A\|^2<2m(2m-5)/(4m-1),$$

then M is totally geodesic.

In the rest part of this section, we shall study a Sasakian space form $M(k)$ in a unit sphere $S^{4m+3}(1)$.

THEOREM 4.3. *Let M be a $(2m+1)$-dimensional 2-totally real and 1-invariant submanifold in a unit sphere $S^{4m+3}(1)$. If M is a Sasakian space form $M(k)$, then M is totally geodesic.*

Proof. Since the ambient space is of constant curvature 1 and the submanifold is a Sasakian space form $M(k)$, we have

$$\sum_a \langle A^aA^a(X),\ Y\rangle=\frac{(m+1)(1-k)}{2}\langle X,\ Y\rangle+\frac{(m+1)(k-1)}{2}\gamma(X)\gamma(Y)$$

by virtue of (1.2) and (2.4).

From this equation and (4.1) we get

(4.4)

$$\begin{aligned}\langle A\circ\tilde{A},\ A\rangle &=\sum_{a,b}(\mathrm{Tr}\,A^aA^b)^2\\ &=\sum_{a,b}\sum_I \langle A^aA^a(e_I),\ A^bA^b(e_I)\rangle=\frac{(m+1)(1-k)}{2}\|A\|^2.\end{aligned}$$

Next, from (1.2) and (2.1) we obtain

$$(k-1)\langle A^N(X), Y\rangle = \sum_a \sum_I [\langle A^a A^N A^a(X), Y\rangle - $$
$$-\langle A^a(X), Y\rangle\langle A^N A^a(e_I), e_I\rangle].$$

Hence

$$(k-1)\|A\|^2 = \sum_{a,b} \sum_I \langle A^a A^b A^a A^b(e_I), e_I\rangle - \langle A\circ\tilde{A}, A\rangle. \tag{4.5}$$

On the other hand, we have

$$\langle \underset{\sim}{A}\circ A, A\rangle = \sum_{a,b} \|[A^a, A^b]\|^2 = 2\langle A\circ\tilde{A}, A\rangle - 2\langle A^a A^b A^a A^b(e_I), e_I\rangle.$$

Substituting (4.5) into this equation, we obtain

$$\langle \underset{\sim}{A}\circ A, A\rangle = -2(k-1)\|A\|^2. \tag{4.6}$$

Therefore, from (4.4) and (4.6) we obtain

$$\langle \underset{\sim}{A}\circ A + A\circ\tilde{A}, A\rangle = \frac{(m+5)(1-k)}{2}\|A\|^2.$$

Hence we have

$$\langle \nabla A, \nabla A\rangle - 4\|A\|^2 = \frac{1}{2}\|A\|^2\big((m-5)(1-k) - 4m - 10\big). \tag{4.7}$$

Since $k > 0$, from Lemma 4.2 and (4.7) we have our assertion.

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