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# HOMEOMORPHISMS HOMOTOPIC TO INDUCED HOMEOMORPHISMS OF WEAK SOLENOIDAL SPACES

BY

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**1. Introduction.** We are concerned with the problem of relating an arbitrary homeomorphism between weak solenoidal spaces to an induced homeomorphism between the spaces. Our main result is that for a certain class of spaces every homeomorphism is homotopic to an induced homeomorphism. This leads one to consider the structure of induced maps between weak solenoidal spaces, a problem treated in the first part of this paper.

We follow the notation of [1] for inverse limit sequences. We denote the projection maps of an inverse limit sequence $(X, f)$ by $f_n: X_\infty \to X_n$.

A *weak solenoidal sequence* $(X, f)$ is an inverse limit sequence where each factor space is a closed, connected, triangulable manifold and each bonding map is a (non-trivial) covering map. The limit space $X_\infty$ is called a *weak solenoidal space.* A map $\varphi: X_\infty \to Y_\infty$ between two weak solenoidal spaces (where $Y_\infty = \lim(Y, g)$) is said to be an *induced map* if there is an order-preserving function $\lambda: Z^+ \to Z^+$ ($Z^+$ denotes the positive integers) and a sequence of maps $\{\varphi_n: X_{\lambda(n)} \to Y_n\}$ with $\varphi_m f_{\lambda(m)}^{\lambda(n)} = g_m^n \varphi_n$ (if $m \leqslant n$) such that $\varphi$ is defined by $g_m \varphi = \varphi_m f_{\lambda(m)}$. In the proofs of section 4, we lose no generality and avoid clumsy notation by assuming $\lambda$ is the identity map.

We show that the set of all induced homeomorphisms on a weak solenoidal space actually forms a subgroup of the group of all homeomorphisms on the space. As an application of this result, we characterize the sequences of maps $\{\varphi_i: X_{\lambda(i)} \to X_i\}$ that induce homeomorphisms on $X_\infty$. The characterizing property is that each $\varphi_{n(i)}$ of some cofinal sequence $\{\varphi_{n(i)}\}$ is a factor of the bonding map $f_{n(i-1)}^{n(i)}$.

In section 5 we consider a special class of weak solenoidal spaces, $D$-like spaces. In this class we find included all weak solenoidal spaces that are inverse limits of either closed, orientable 2-manifolds or "sufficiently large" (see [8]), irreducible, orientable, closed 3-manifolds. We show that any homeomorphism between $D$-like spaces is homotopic to an induced homeomorphism.

**2. Preliminaries.** In this section we record some results concerning maps between weak solenoidal spaces that we will use quite strongly in the sequel. Stronger versions of these two theorems are proved in [5].

THEOREM 2.1. *Let $\varepsilon > 0$ and let $F$ be a map between weak solenoidal spaces. Then there is an induced map between the spaces that is $\varepsilon$-homotopic to $F$.*

A map $F : X_\infty \to Y_\infty$ is *fiber-preserving* if there is an integer $n$ such that for each $x \in X_n$ there is a $y \in Y_1$ such that $F(f_n^{-1}(x)) \subset g_1^{-1}(y)$.

THEOREM 2.2. *A map between weak solenoidal spaces is an induced map if and only if it is fiber-preserving.*

Let $\lambda : Z^+ \to Z^+$ be a strictly increasing function and $(X, f)$ a weak solenoidal sequence. A sequence $\{\varphi_i : X_{\lambda(i)} \to X_i\}$ of maps is said to be *factorable* (with respect to $(X, f)$) if there is a cofinal subsequence $\{\varphi_{n(j)}\}$ such that $\varphi_{n(j)}$ is a factor of $f_{\lambda(n(j-1))}^{\lambda(n(j))}$ (that is, $\psi_j \cdot \varphi_{n(j)} = f_{\lambda(n(j-1))}^{\lambda(n(j))}$).

**3. Models for path components.** When considering a map between weak solenoidal spaces, it is quite often sufficient to consider only its restriction to one path component. In many situations the fact that the path component is not locally path-connected tends to obscure the problem, so these models will facilitate a lucid argument which would otherwise be rather cumbersome.

Let $K$ be a path component of the weak solenoidal space $X_\infty = \lim(X, f)$. The *model* $\hat{K}$ for $K$ is the set $K$ retopologized by the topology generated by path components of open sets of $K$ (where $K$ has the relative topology induced by $X_\infty$). Thus $\hat{K}$ is locally path-connected. Moreover, each projection $f_n : X_\infty \to X_n$, when restricted to $K$, defines a covering map $\hat{f}_n : \hat{K} \to X_n$. (Models from a slightly different viewpoint are discussed in [6].)

LEMMA 3.1. *Let $(X, f)$ and $(Y, g)$ be weak solenoidal sequences and let $F : X_\infty \to Y_\infty$ be a continuous map. Let $K$ be a path component of $X_\infty$, and let $L$ be the path component of $Y_\infty$ containing $F(K)$. Given an $\varepsilon > 0$, if there is a map $\varphi : X_n \to Y_m$ (for some $m, n$) such that the diagram*

$$\begin{array}{ccc} X_n & \xleftarrow{\ \hat{f}_n\ } & \hat{K} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \hat{F}} \\ Y_m & \xleftarrow{\ \hat{g}_m\ } & \hat{L} \end{array}$$

*is $\varepsilon/3$-commutative, then the diagram*

$$\begin{array}{ccc} X_n & \xleftarrow{\ f_n\ } & X_\infty \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle F} \\ Y_m & \xleftarrow{\ g_m\ } & Y_\infty \end{array}$$

*is $\varepsilon$-commutative. Hence if $\varphi\hat{f}_n = \hat{g}_m\hat{F}$, then $\varphi f_n = g_m F$.*

Proof. Since $\varphi f_n$ and $g_m F$ are uniformly continuous, there is a $\delta > 0$ such that for any $A \subset X_\infty$ with $\operatorname{diam}(A) < \delta$ we infer that $\operatorname{diam}(\varphi f_n(A)) < \varepsilon/3$ and $\operatorname{diam}(g_m F(A)) < \varepsilon/3$.

Let $x$ be an arbitrary point of $X_\infty$. Choose some $y \in K$ such that $d(x, y) < \delta$ ($K$ is dense in $X_\infty$).

Then

$$d(\varphi f_n(x), g_m F(x)) < \varepsilon/3 + d(\varphi f_n(y), g_m F(y)) + \varepsilon/3 \leqslant \varepsilon.$$

**4. The group of induced homeomorphisms.** Let $G(X_\infty)$ denote the group of all self-homeomorphisms of the weak solenoidal space $X_\infty$. In this section we show that the subset $H(X_\infty)$ of $G(X_\infty)$ consisting of all the induced homeomorphisms is actually a subgroup. Before obtaining this result, we prove the following interesting and very natural lemma concerning the nature of a sequence of maps (between the factor spaces) that induces a local homeomorphism.

LEMMA 4.1. *If $\varphi$ is an induced local homeomorphism between weak solenoidal spaces and if $\{\varphi_i\}$ is a sequence of maps inducing $\varphi$, then each $\varphi_i$ is a covering map.*

Proof. Let $X_\infty$ and $Y_\infty$ be weak solenoidal spaces, and let $\varphi : X_\infty \to Y_\infty$ be a local homeomorphism induced by the sequence $\{\varphi_i : X_i \to Y_i\}$. Let $K$ be a path component of $X_\infty$ and let $L = \varphi(K)$, a path-component of $Y_\infty$. Since $\varphi_1 f_1 = g_1\varphi$, we have the following commutative diagram involving the models $\hat{K}, \hat{L}$ for $K, L$, respectively,

$$\begin{array}{ccc} X_1 & \xleftarrow{\ \hat{f}_1\ } & \hat{K} \\ \Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \hat{\varphi}} \\ Y_1 & \xleftarrow[\ \hat{g}_1\ ]{} & \hat{L} \end{array}$$

where $\hat{f}_1$ and $\hat{g}_1$ are covering maps and $\hat{\varphi}$ is a local homeomorphism.

We observe that $\varphi_1$ is a local homeomorphism between compact polyhedral manifolds and hence a finite-to-one covering map. Since $\varphi_1 \circ f_1^m = g_1^m \circ \varphi_m$, we see that $\varphi_m$ is a lifting of the covering map $\varphi_1 \circ f_1^m$ and thus also a covering map for each $m$.

COROLLARY 4.2. *Every induced homeomorphism between weak solenoidal spaces is induced by covering maps.*

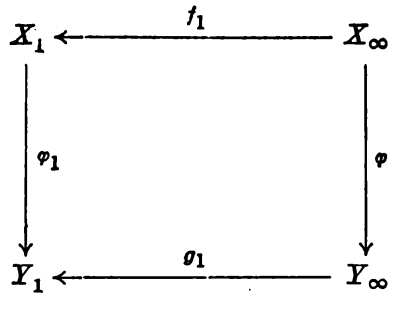
Remark 4.3. This corollary is the strongest possible in that the inducing maps need not be homeomorphisms. For example, consider any weak solenoidal space $(X, f)$ with all factor spaces homeomorphic and only a single bonding map $f$. Then the sequence $\{f: X_{i+1} \to X_i\}$ of non-trivial covering maps induces a homeomorphism on $X_\infty$.

In section 5 we show that arbitrary homeomorphisms on weak solenoidal spaces are $\varepsilon$-homotopic to induced maps such that each member of the inducing sequence $\{\varphi_i : X_{n(i)} \to X_i\}$ induces an injection of $\pi_1(X_{n(i)})$ into $\pi_1(X_i)$, and for certain classes of manifolds, the $\varphi_i$ are actually covering maps.

THEOREM 4.4. *The set $H(X_\infty)$ of all induced self-homeomorphisms of $X_\infty$ is a subgroup of the group $G(X_\infty)$ of all self-homeomorphisms of $X_\infty$.*

Proof. Let $X_\infty = \lim(X, f)$, $Y_\infty = \lim(Y, g)$, and $\varphi : X_\infty \to Y_\infty$ be a homeomorphism induced by the sequence $\{\varphi_i : X_i \to Y_i\}$ of maps. To prove this theorem, it suffices to show that $\varphi^{-1}$ is an induced homeomorphism.

According to Corollary 4.2, each $\varphi_i$ is a covering map. It is helpful to keep the following commutative diagram in mind during the next argument, which shows that $\varphi^{-1}$ is a fiber-preserving map, and hence, by Theorem 2.2, an induced map:



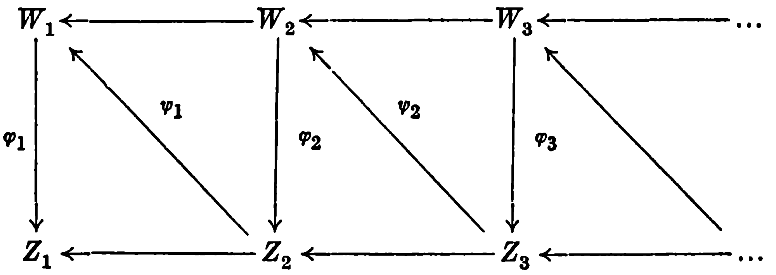

Let $V$ and $W$ be open $n$-cells in $Y_1$ with $\overline{W} \subset V$ ($\overline{W}$ denotes the closure of $W$). Let $S$ be one of the finite number of open cells in $X_1$ evenly covering $V$ and let $R \subset S$ cover $W$. The set $\varphi f_1^{-1}(\bar{R})$ is a compact subset of $Y_\infty$ which is contained in the open set $\varphi f_1^{-1}(S)$. Hence $\varphi f_1^{-1}(S)$ is the union of basic open sets of $Y_\infty$, and the compact set $\varphi f_1^{-1}(\bar{R})$ is contained in a finite union of these basic open sets. It follows that there exists an integer $j$ and a finite collection $\{O_1, \ldots, O_k\}$ of open sets in $Y_j$ such that

$$\varphi f_1^{-1}(\bar{R}) \subset \bigcup [g_j^{-1}(O_i) : i = 1, \ldots, k] \subset \varphi f_1^{-1}(S).$$

Then for any point $z_j$ in $\bigcup\{O_i\} \subset Y_j$ such that $z_1 = g_1^j(z_j) \in W$, we have $(g_j\varphi)^{-1}(z_j) \subset (g_1\varphi)^{-1}(z_1) = (\varphi_1 f_1)^{-1}(z_1) = f_1^{-1}(r)$, where $r$ is the point of $R$ which covers $z_1$. Finally, since $X_1$ is covered by a finite number of open sets such as $R$, there is an index $\lambda$ such that for each $z$ in $Y_\lambda$, there is an $x \in X_1$ such that $\varphi^{-1}(g_\lambda^{-1}(z)) \subset f_1^{-1}(x)$. Therefore $\varphi^{-1}$ is fiber--preserving.

Remark. $H(X_\infty)$ is not a normal subgroup of $G(X_\infty)$, as one can easily find an induced homeomorphism $f: X_\infty \to X_\infty$ and a non-fiber--preserving homeomorphism $g: X_\infty \to X_\infty$ such that $gfg^{-1}$ is not fiber--preserving.

COROLLARY 4.5. *If $\varphi: X_\infty \to X_\infty$ is an induced homeomorphism, then there are subsequences $\{W_i\}$ and $\{Z_i\}$ of $\{X_i\}$ such that the following diagram commutes and all maps are covering maps:*




As an application of Theorem 4.4, we are able to obtain the following characterization of sequences of maps that induce homeomorphisms:

THEOREM 4.6. *A sequence of maps induces a homeomorphism between weak solenoidal spaces if and only if the sequence is factorable.*

A routine exercise in the application of Corollary 4.5 yields this theorem. We omit the details.

**5. $D$-like spaces.** Let $D$ be the class of all closed manifolds $M$ with the following property:

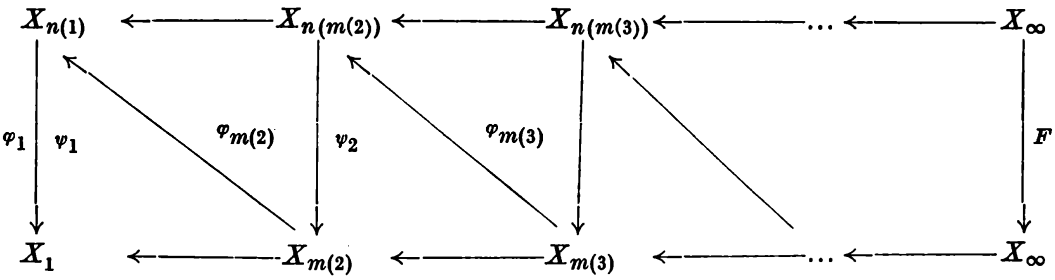
If $f: M \to N$ is a map of $M$ into another closed manifold $N$ of the same dimension such that the induced homomorphism $f_\#: \pi_1(M) \to \pi_1(N)$ is an injection, then $f$ is homotopic to a covering projection of $M$ onto $N$.

It is known (see [8]) that all closed, orientable 2-manifolds belong to the class $D$. More recently, Waldhausen [8] has proved that a large class of closed 3-manifolds belong to $D$, namely, all closed, irreducible, orientable 3-manifolds that are sufficiently large. Heil [4] extended this to the non-orientable, $P^2$-irreducible, closed 3-manifolds that are sufficiently large.

If $X_\infty = \lim(X, f)$ is a weak solenoidal space with each factor space $X_n \in D$, we say that $X_\infty$ is *$D$-like*. We show that any homeomorphism $F$ on a $D$-like weak solenoidal space is homotopic to an induced homeomorphism that agrees with $F$ on any desired finite set of points.

The first lemma of this section applies to general weak solenoidal sequences and points out the reason for considering $D$-like weak solenoidal spaces.

LEMMA 5.1. *Given any $\varepsilon > 0$ and an arbitrary homeomorphism $F: X_\infty \to X_\infty$, there exists a sequence of maps $\{\varphi_i : X_{n(i)} \to X_i\}$ inducing a map $\varphi : X_\infty \to X_\infty$ that is $\varepsilon$-homotopic to $F$. Furthermore, there is another sequence of maps $\{\psi_i : X_{m(i)} \to X_{n(m(i))}\}$ such that in the diagram*




*each parallelogram is commutative and each triangle is $\varepsilon$-commutative.*

Proof. Let $\varepsilon > 0$ be small enough so that any two maps into $X_1$ with distance between them less than $\varepsilon$ are homotopic. Then by Theorem 2.1 we have a sequence of maps $\{\varphi_i : X_{n(i)} \to X_i\}$ that induces a map $\varphi : X_\infty \to X_\infty$ with $\varphi \simeq F$ (by an $\varepsilon/2$-homotopy) and such that $d(\varphi_i f_{n(i)}, f_i F) < \varepsilon/2$. Now choose $\delta > 0$ such that if $A \subset X_{n(i)}$ has diameter $\leqslant \delta$, then $\operatorname{diam}(\varphi_i(A)) < \varepsilon/2$ (for any $i$). Apply Theorem 2.1 again, this time to $F^{-1}$, to obtain a sequence of maps $\{\psi_i : X_{m(i)} \to X_{n(i)}\}$ such that $d(\psi_i f_{m(i)}, f_{n(i)} F^{-1}) < \delta$.

COROLLARY 5.2. *If $F : X_\infty \to X_\infty$ is a homeomorphism and $\varepsilon > 0$, then there is a sequence of maps $\{\varphi_i : X_{n(i)} \to X_i\}$ inducing a map $\varphi : X_\infty \to X_\infty$ that is $\varepsilon$-homotopic to $F$ and such that each $\varphi_i$ induces a monomorphism $(\varphi_i)_\#$ between the fundamental groups of $X_{n(i)}$ and $X_i$ for all $i \geqslant m(2)$.*

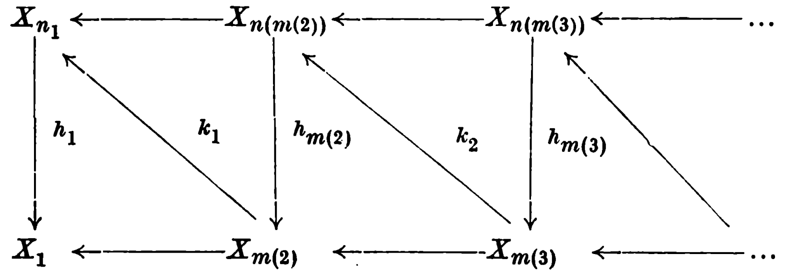
Proof. Let $\{\varphi_i : X_{n(i)} \to X_i\}$ be a sequence as in the proof of Lemma 5.1. Then there is a map $\psi_1 : X_{m(2)} \to X_{n(1)}$ such that $d(\psi_1 \cdot \varphi_{m(2)}, f_{n(1)}^{n(m(2))}) < \varepsilon$. Hence $\psi_1 \cdot \varphi_{m(2)} \simeq f_{n(1)}^{n(m(2))}$. But $f_1^{m(2)}$ is a covering map; thus $\psi_1 \cdot \varphi_{m(2)}$ induces an injection on the fundamental group, and hence so does $\varphi_{m(2)}$. For each $i > m(2)$, $\varphi_i$ is a lifting of $\varphi_{m(2)} \cdot f_{m(2)}^i$ and therefore also induces an injection.

THEOREM 5.3. *Let $X_\infty$ be a $D$-like weak solenoidal space and $F : X_\infty \to X_\infty$ any homeomorphism. Then there is an induced homeomorphism $\varphi$ that is homotopic to $F$.*

Proof. Let $\varepsilon > 0$ be so small that any two maps into $X_1$ that are within $\varepsilon$ of each other are homotopic. Let $\{\varphi_i : X_{n(i)} \to X_i\}$ be a sequence of maps as in Lemma 5.1, inducing a map $\varphi$ $\varepsilon$-homotopic to $F$. Then $\varphi_{m(2)}$ induces a monomorphism $(\varphi_{m(2)})_\# : \pi_1(X_{n(m(2))}) \to \pi_1(X_{m(1)})$. Since $X_{n(m(2))} \in D$, there is a covering map $h_{m(2)} : X_{n(m(2))} \to X_{m(2)}$ homotopic to

$\varphi_{m(2)}$. Using the homotopy lifting property, we get a sequence of homotopies $\{H_i : \varphi_i \simeq h_i\}$ (for each $i \geqslant m(2)$) such that $H_i f_{n(i)}^{n(i+1)} = f_i^{i+1} H_{i+1}$ and $h_i$ is a covering map. Hence the maps $h_i$ induce a local homeomorphism $h: X_\infty \to X_\infty$ onto $X_\infty$ that is homotopic to $F$. All we need to do is check that $h$ is one-to-one, and thus a homeomorphism.

We can construct a sequence $\{k_i \simeq \psi_i\}$ of covering maps such that the following diagram commutes, since $f_1^{m(2)} \simeq h_1 \circ \psi_1$:




This shows that $h$ is a homeomorphism.

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