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SOME CLASSES OF LOCALLY CONNECTED CONTINUA

BY

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Introduction. The first part of the paper contains the generalization of Whyburn's theorem about trees and light open mappings to the nonmetric case. In connection with it several questions are stated with some solutions for metric continua.

Next we construct two examples. First one solves Lelek's problem by showing the existence of a unicoherent continuum which has rim-type 2 and which contains a non-arcwise connected subcontinuum. The second example shows that there is a rim-finite metric space which has exactly one hereditarily locally connected compactification.

1. Preliminaries. All spaces considered are Hausdorff and compact and mappings are continuous surjections. A mapping f from X onto Y is

(i) *open* if f maps every set open in X onto a set open in Y;

(ii) *confluent* if for each continuum K in Y, each component of $f^{-1}(K)$ is mapped onto K by f (see [1], p. 213);

(iii) *locally confluent* if for each point y of Y there is an open set U containing y such that $f|f^{-1}(\mathrm{cl}(U))$ is confluent (see [7], p. 106);

(iv) *light* if each point inverse is totally disconnected (see [16], p. 130).

The next two propositions are well known in the metric case (see [16], p. 148, and [7], p. 109).

PROPOSITION 1. *Every open mapping is confluent.*

Indeed, if K is a subcontinuum of Y, then $f|f^{-1}(K)$ is open. Therefore, if U is an open-closed set in $f^{-1}(K)$, then $f(U)$ is an open-closed set in K; thus $f(U) = K$. Hence, if Q is a component of $f^{-1}(K)$, then it is a quasi-component of $f^{-1}(K)$ and $f(Q) = K$.

PROPOSITION 2. *If f: $X \to Y$ is a light locally confluent mapping from X onto a locally connected space Y, then f is open.*

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In fact, let U be an open set in X and $x \in U$. It suffices to show that $f(x) \in \operatorname{int} f(U)$. Since $f^{-1}f(x)$ is totally disconnected, there are two open and disjoint sets G and H such that $f^{-1}f(x) \subset G \cup H$ and $x \in G \subset \operatorname{cl}(G) \subset U$. Since $f(\operatorname{cl}(G) \setminus G)$ is a closed set which does not contain $f(x)$ and f is locally confluent, we find an open connected set V such that $f(x) \in V \subset \operatorname{cl}(V) \subset Y \setminus f(\operatorname{cl}(G) \setminus G)$ and $f \mid f^{-1}(\operatorname{cl}(V))$ is confluent. Let K be the component of $f^{-1}(\operatorname{cl}(V))$ containing x. Then $f(K) = \operatorname{cl}(V)$. Moreover, since $f^{-1}(\operatorname{cl}(V)) \subset X \setminus (\operatorname{cl}(G) \setminus G)$, we infer that $K \subset G$. Therefore, $f(x) \in V \subset \operatorname{cl}(V) = f(K) \subset f(G) \subset f(U)$; thus $f(x) \in \operatorname{int} f(U)$.

We have the following characterization of light mappings (compare [16], p. 131; the proof is omitted, because it is simple and technical, but long).

PROPOSITION 3. *The mapping $f\colon X \to Y$ is light if and only if for every finite open cover $\mathscr{G}$ of X there is a finite open cover $\mathscr{H}$ of Y such that if K is a subcontinuum of Y which is contained in some element of $\mathscr{H}$, then each component of $f^{-1}(K)$ is contained in some element of $\mathscr{G}$.*

Recall that a *generalized arc* is a continuum with exactly two non-cut-points. A *tree* is a continuum in which each pair of distinct points can be separated by some third point. Let X be a tree and let p be an arbitrary fixed point in X. The partial order $\leqslant_p$ is defined as follows: $x, y \in X$; $x \leqslant_p y$ if and only if the pair p, y is separated by x or $x = y$ or $x = p$. Then $x <_p y$ means that $x \leqslant_p y$ and $x \neq y$. If $M_p(x) = \{y \in X\colon x \leqslant_p y\}$, then (see [9], Lemma 2.8; [11], Lemma 4; [14] and [15]).

PROPOSITION 4. *If a continuum X is a tree, then*

(i) *$M_p(x)$ is closed and $M_p(x) \setminus \{x\}$ is open for every $x \in X$;*

(ii) *the multifunction $M_p\colon X \to X$ is upper semi-continuous;*

(iii) *if $A \subset X$ is connected and $A \cap M_p(x) \neq \emptyset \neq A \cap (X \setminus M_p(x))$, then $x \in A$;*

(iv) *every nonempty closed subset of X contains a maximum element in $\leqslant_p$.*

Recall that if n is a cardinal number $\leqslant \mathfrak{c}$ or the ordinal number ω, the space X is said to be of *order $\leqslant n$ at the point p*,

$$\operatorname{ord}_p X \leqslant n,$$

if for every neighbourhood G of p there exists an open set H such that $p \in H \subset G$ and $\operatorname{card}(\operatorname{bd}(H)) \leqslant n$. The points of order $\leqslant \omega$ are said to be *regular* (these are the points for which the set $\operatorname{bd}(H)$ is finite). A space which consists exclusively of regular points is said to be *regular* or *rim-finite*. It easily follows from Proposition 4 that

PROPOSITION 5. *Every tree is rim-finite.*

A continuum is said to be *rational* if it has a basis of neighbourhoods with countable boundaries. A continuum is said to be *finitely Suslinian* if there do not exist an open cover $\mathscr{G}$ of X and an infinite family $\mathscr{A}$ of pairwise disjoint subcontinua of X such that for each $A \in \mathscr{A}$, $A \not\subset G$ for any $G \in \mathscr{G}$ (see [13]). A continuum is said to be *hereditarily locally connected* if every subcontinuum is locally connected. Every 1-dimensional continuum is called a *curve*.

The following diagram gives the known inclusions among the above classes of continua (compare [13], Lemma 2 and Corollary 4):

$$\begin{tikzcd}
\text{arcs} \arrow[r] & \text{trees} \arrow[r] & \text{rim-finite continua} \arrow[r] \arrow[ddr] & \text{finitely Suslinian continua} \arrow[r] & \text{hereditarily locally connected continua} \arrow[d] \\
 & & & & \text{arcwise connected curves} \arrow[d] \\
 & & & \text{rational continua} \arrow[r] & \text{curves}
\end{tikzcd}$$

In the metric case it is known (see [16], p. 94) that every hereditarily locally connected continuum is rational. We do not know the answer to the following:

Problem 1. Does there exist a hereditarily locally connected continuum which is not rational? (**P 1303**)

All locally connected metric continua are images of the simple closed interval (see [5], p. 256); thus they are arcwise connected (see [16], p. 36). In the nonmetric case we have a more complicated situation (see [12] for open questions and for references of partial results). In particular we have

Problem 2. Is it true that every locally connected rational continuum is arcwise connected? (**P 1304**)

2. Whyburn's theorem. We will generalize Whyburn's theorem (see [16], p. 186) to the nonmetric case (see [3] and [10] for other generalizations). Throughout this section the results that are used to manipulate nets and nets of sets are found in [2], [4] and [8].

Let A be a finite subset of a tree X with $p \in A$. We put

$$M_p(x, A) = M_p(x) \setminus (\bigcup \{M_p(x^\gamma) \setminus \{x^\gamma\}: x <_p x^\gamma \in A\}).$$

From Proposition 4 we obtain

Proposition 6. *$M_p(x, A)$ is a continuum and $X = \bigcup \{M_p(x, A): x \in A\}$.*

Theorem 1 (Whyburn). *Let X be a tree, f be an open light mapping from a space Y onto a space Z, $z_0 \in Z$ and let $y_0 \in f^{-1}(z_0)$. If g is a mapping from X onto Z and $p \in g^{-1}(z_0)$, then there exists a mapping h from X into Y such that $g = f \circ h$ and $h(p) = y_0$.*

Proof. Let Σ denote the collection of all finite subsets of X which contain the point p. The set Σ is directed by inclusion. If $A \in \Sigma$, then $A = \{x_0^A, x_1^A, \ldots, x_{n_A}^A\}$. We can assume that $x_0^A = p$ and

$$\Big(\bigcup_{i=0}^{k-1} M_p(x_i^A, A)\Big) \cap M_p(x_k^A, A) \neq \emptyset \quad \text{for} \quad k = 1, 2, \ldots, n_A.$$

Since f is confluent (by Proposition 1), we can find continua $K(x, A)$ in Y such that

(1) $\quad y_0 \in K(x_0^A, A) \quad$ and $\quad \Big(\bigcup_{i=0}^{k-1} K(x_i^A, A)\Big) \cap K(x_k^A, A) \neq \emptyset$

for $k = 1, 2, \ldots, n_A$,

(2) $\quad f(K(x, A)) = g(M_p(x, A)) \quad$ for $\quad x \in A$.

Consider a net $\mathscr{S} = \{S_A;\ A \in \Sigma\}$ where continua S_A are defined by

$$S_A = \bigcup \{M_p(x, A) \times K(x, A):\ x \in A\}.$$

Since the space $2^{X \times Y}$ is compact, this net has a cluster point S. Therefore S is a limit of a net $\mathscr{S}^0 = \{S_B^0;\ B \in \Sigma^0\}$ which is finer than the net $\mathscr{S}$, i.e., there exists a function $\varphi\colon \Sigma^0 \to \Sigma$ such that

(3) for every $A \in \Sigma$ there exists a $B \in \Sigma^0$ such that $\varphi(B') \geqslant A$ for $B \leqslant B' \in \Sigma^0$,

(4) $\quad S_{\varphi(B)} = S_B^0 \quad$ for $\quad B \in \Sigma^0$.

Let $S(x) = (\{x\} \times Y) \cap S$ for $x \in X$ and let π denote the natural projection from $X \times Y$ onto Y. We define

$$h(x) = \pi(S(x)) \quad \text{for} \quad x \in X.$$

Since the graph of h is equal to the set S which is compact in $X \times Y$, we infer that h is an upper semi-continuous mapping (in the sequel we will show that h is single-valued and therefore h will be a continuous mapping). Now we will prove

(5) $\quad$ if $(x, y) \in S(x)$, then $g(x) = f(y)$.

In fact, let $(x, y) \in S(x)$ and suppose, on the contrary, that $g(x) \neq f(y)$. Since Z is a Hausdorff space, there are open disjoint neighbourhoods G and F of $g(x)$ and $f(y)$, respectively. Since g is continuous and X is rim-finite (Proposition 5), we find an open connected neighbourhood U of x in X such that the boundary of U ($\mathrm{bd}(U)$) is finite and $g(\mathrm{cl}(U)) \subset G$. Since f is continuous, we can find an open neighbourhood V of y in Y such that $f(V) \subset F$.

Consider $\Sigma^1 = \{B \in \Sigma^0\colon \mathrm{bd}(U) \subset \varphi(B)\}$. Since Σ^1 is cofinal with Σ^0, we have $S = \lim \{S_B^0;\ B \in \Sigma^1\}$. Take $H = \{C \in 2^{X \times Y}\colon C \cap (U \times V) \neq \emptyset\}$. Since H

is an open set in $2^{X\times Y}$ containing S, there is $B\in\Sigma^1$ such that $S^0_{B'}\in H$ for $B\leqslant B'\in\Sigma^1$.

By the definition of $S_{\varphi(B)}$ and equality (4) we conclude that

$$\big(M_p(x_k^{\varphi(B)},\ \varphi(B))\times K(x_k^{\varphi(B)},\ \varphi(B))\big)\cap(U\times V)\neq\emptyset$$

$$\text{for some } k=0,1,\ldots,n_{\varphi(B)}.$$

In particular $M_p(x_k^{\varphi(B)},\varphi(B))\cap U\neq\emptyset$; but then $M_p(x_k^{\varphi(B)},\varphi(B))\subset \mathrm{cl}(U)$ because $\mathrm{bd}(U)\subset\varphi(B)$. Therefore

$$g\big(M_p(x_k^{\varphi(B)},\ \varphi(B))\big)=f\big(K(x_k^{\varphi(B)},\ \varphi(B))\big)\subset G$$

by (2). Since $K(x_k^{\varphi(B)},\varphi(B))\cap V\neq\emptyset$ and $f(V)\subset F$, we obtain

$$f\big(K(x_k^{\varphi(B)},\ \varphi(B))\big)\cap F\neq\emptyset;$$

thus $G\cap F\neq\emptyset$, a contradiction.

We claim that

(6) the set $S(x)$ is degenerate for each $x\in X$.

Indeed, it follows from (5) that $S(x)$ is totally disconnected, because f is light. Suppose, on the contrary, that (x,y), $(x,y')\in S(x)$ and $y\neq y'$. Since Z is locally connected (as a continuous image of a locally connected compact space), there is an open connected neighbourhood U of $g(x)(=f(y)=f(y'))$ such that $f^{-1}(\mathrm{cl}(U))$ is not connected between y and y'. Therefore there are two disjoint closed sets P and Q such that $y\in P$, $y'\in Q$ and $f^{-1}(\mathrm{cl}(U))=P\cup Q$. Take an open connected neighbourhood V of x with a finite boundary and such that $g(\mathrm{cl}(V))\subset U$. Then

$$f^{-1}(g(V))=\big(f^{-1}(g(V))\cap P\big)\cup\big(f^{-1}(g(V))\cap Q\big).$$

Consider $\Sigma^2=\{B\in\Sigma^0\colon \mathrm{bd}(V)\subset\varphi(B)\}$. Since Σ^2 is cofinal with Σ^0 we obtain $S=\lim\{S^0_B;\ B\in\Sigma^2\}$. Take

$$H=\big\{C\in 2^{X\times Y}\colon C\cap\big(V\times(f^{-1}(U)\cap P)\big)\neq\emptyset \text{ and }$$
$$C\cap\big(V\times(f^{-1}(U)\cap Q)\big)\neq\emptyset\big\}.$$

Since H is an open set in $2^{X\times Y}$ containing S, there is $B\in\Sigma^2$ such that $S^0_{B'}\in H$ for $B\leqslant B'\in\Sigma^2$.

Since $\mathrm{bd}(V)\subset\varphi(B)$ and V is connected, we infer that the set

$$N=\bigcup\{M_p(x,\varphi(B))\times K(x,\varphi(B))\colon\ x\in\varphi(B)\ \text{ and }\ M_p(x,\varphi(B))\cap V\neq\emptyset\}$$

is connected. Therefore the relations

$$N=S_{\varphi(B)}\cap\big(\mathrm{cl}(V)\times f^{-1}(\mathrm{cl}(U))\big)\subset(\mathrm{cl}(V)\times P)\cup(\mathrm{cl}(V)\times Q)$$

imply either $S_{\varphi(B)}\cap(V\times P)=\emptyset$ or $S_{\varphi(B)}\cap(V\times Q)=\emptyset$, a contradiction, because $S_{\varphi(B)}=S^\Gamma_B\in H$. This completes the proof of Theorem 1.

The main idea of our above proof is the same as in the metric case, but the details are quite different. From Theorem 1 we conclude

COROLLARY 1. *Let T be a tree (arc) contained in a space Y and let $p \in T$. If f is a locally confluent light mapping from a space X onto Y and $x_0 \in f^{-1}(p)$, then there is a continuum X_0 in X containing x_0 such that $f|X_0$ is a homeomorphism from X_0 onto Y.*

Since every metric locally connected continuum is an image of an arc, we obtain, by Proposition 2 and Theorem 1,

COROLLARY 2. *If f is a locally confluent light mapping from a compact space onto a locally connected metric continuum Y, then there is a locally connected metric continuum X_0 in X such that $f(X_0) = Y$.*

The following questions remain open:

PROBLEM 3. Can the assumption of metrizability of Y in Corollary 2 be omitted? **(P 1305)**

PROBLEM 4. Is it true that if f is an open light mapping from a continuum X onto a continuum Y which is rim-finite (resp. finitely Suslinian, hereditarily locally connected, rational and locally connected), then there is a continuum X_0 in X such that $f(X_0) = Y$ and X_0 is rim-finite (resp. finitely Suslinian, hereditarily locally connected, rational and locally connected)? **(P 1306)**

We will give some positive answers for Problem 4 in the metric case in the next section. This problem for arcwise connected continua has a negative answer. Namely, we have

Example 1. Let

$$S = \{(x, \sin(\pi/x)):\ 0 < x \leqslant 1\} \cup \{(0, y):\ -1 \leqslant y \leqslant 1\}.$$

If we identify the point $(0, 0, 0)$ with $(1, 0, 1)$ and the point $(1, 0, 0)$ with $(0, 0, 1)$ in the product $S \times \{0, 1\}$, then we obtain a continuum X which can be mapped by a two-to-one (antipodal) open mapping f onto an arcwise connected continuum Y obtained from S by the identification of the point $(0, 0)$ with the point $(1, 0)$. Moreover, no arcwise connected subcontinuum of X is mapped onto Y under f.

3. Mappings from metric trees.

LEMMA 1. *If U is an open connected subset of a locally connected metric continuum X and* $\dim \mathrm{bd}(U) \leqslant 0$, *then* $\mathrm{cl}(U)$ *is a locally connected continuum.*

Proof. This easily follows from the fact that if $\mathrm{cl}(U)$ is not locally connected at p, then p belongs to a nondegenerate continuum K which consists of points at which $\mathrm{cl}(U)$ is not locally connected (see [5], Theorem 1, p. 245). Since $\dim \mathrm{bd}(U) \leqslant 0$, the continuum K is not contained in $\mathrm{bd}(U)$. Then U must contain a point at which it is not locally connected. This is impossible, and the proof is complete.

LEMMA 2. *Let X be a locally connected compact metric space which has a finite number of components and* $\dim X = 1$ *and let* $X = G_1 \cup \ldots \cup G_k$ *be a given open cover of X. Then there exists a system of closed sets $F_1, \ldots, F_k$ satisfying the conditions*

(i) $\quad X = F_1 \cup \ldots \cup F_k, \quad F_i \subset G_i, \quad \dim(F_i \cap F_j) \leqslant 0 \quad$ *for* $i \neq j$

and each F_i has a finite number of components.

Proof. Proceed by induction. For $k = 2$ setting $X = G_1 \cup G_2$, $A = X - G_1$ and $B = X - G_2$ we have $A \cap B = \emptyset$.

Since X is 1-dimensional, each point has small connected neighbourhoods which have zero-dimensional boundaries (compare [5], Theorem 3, p. 238). For each $x \in A$ take an open connected neighbourhood V_x such that $x \in V_x \subset \mathrm{cl}(V_x) \subset X \setminus B$ and $\dim \mathrm{bd}(V_x) \leqslant 0$. Let $V_1, \ldots, V_n$ be a finite cover of A which is formed by such sets. Consider the set $V = \mathrm{cl}(V_1) \cup \ldots \cup \mathrm{cl}(V_n)$. Then V has a finite number of components and $\dim \mathrm{bd}(V) \leqslant 0$. Moreover, since $V \cap B = \emptyset$, the number $\varepsilon = \inf\{d(x, x^1)\colon x \in V, x^1 \in B\}$ is greater than 0, where d is a metric in X. Let $B(V, \varepsilon/2)$ be a generalized ball around the set V with radius $\varepsilon/2$. It follows from Theorem 16 in [5], p. 234, that all components of $X \setminus V$ except a finite number are contained in $B(V, \varepsilon/2)$. Take $G = V \cup \bigcup \{C\colon C$ is a component of X and $C \subset B(V, \varepsilon/2)\}$. Then G and $X \setminus G$ have a finite number of components, $\mathrm{bd}(G) \cup \mathrm{bd}(X \setminus G) \subset \mathrm{bd}(V)$; thus also $\dim \mathrm{bd}(G) = \dim \mathrm{bd}(X \setminus G) \leqslant 0$. Furthermore, $A \subset G$ and $\mathrm{cl}(G) \cap B = \emptyset$. Therefore, the sets $F_1 = \mathrm{cl}(X \setminus G)$ and $F_2 = \mathrm{cl}(G)$ satisfy conditions (i) of this lemma for $k = 2$.

Now assume that the lemma holds for $k-1$. As we have just proved there exist two closed sets H and F_k such that

(1) $\quad X = H \cup F_k, \quad H \subset G_1 \cup \ldots \cup G_{k-1}, \quad F_k \subset G_k, \quad \dim(H \cap F_k) \leqslant 0$
and H and F_k have a finite number of components.

Since the set H is a locally connected (by Lemma 1) compact metric space, which has a finite number of components and $\dim H = 1$, the identity $H = (H \cap G_1) \cup \ldots \cup (H \cap G_{k-1})$ implies by hypothesis the existence of a system of closed sets $F_1, \ldots, F_{k-1}$ satisfying the conditions

(2) $\quad H = F_1 \cup \ldots \cup F_{k-1}$, $F_i \subset H \cap G_i$, $\dim(F_i \cap F_j) \leqslant 0$ for $i < j < k$
and each F_k has a finite number of components.

Conditions (i) follow easily from (1) and (2).

From Lemma 2 we conclude (compare [5], Theorem 11, p. 288)

LEMMA 3. *If X is a 1-dimensional locally connected metric continuum, and A is a 0-dimensional closed subset of X, then for every $\varepsilon > 0$ there exists a finite system of locally connected continua $K_1, \ldots, K_n$ satisfying the conditions*

(i) $\quad X = K_1 \cup \ldots \cup K_n$, $\mathrm{diam}(K_i) < \varepsilon$, $\dim(K_i \cap K_j) \leqslant 0$, $A \cap K_i \cap K_j = \emptyset$
for every system of distinct subscripts.

Proof. Since A is closed and 0-dimensional we can find a finite system of open sets $V_1, \ldots, V_m$ such that $A \subset V_1 \cup \ldots \cup V_m$, $\mathrm{cl}(V_i) \cap \mathrm{cl}(V_j) = \emptyset$ for $i \neq j$ and $\operatorname{diam}\mathrm{cl}(V_i) < \varepsilon/3$. If $\delta = \inf\{d(x, x^\nabla): x \in \mathrm{cl}(V_i), x^\nabla \in \mathrm{cl}(V_j), i \neq j$ and $i, j = 1, \ldots, m\}$, then $\delta > 0$. Consider $0 < \eta < \min\{\delta/3, \varepsilon/3\}$. Since $\dim X = 1$, there exists a system of open sets $G_1, \ldots, G_k$ satisfying the conditions $X = G_1 \cup \ldots \cup G_k$, $\operatorname{diam}(G_i) < \eta$, $G_h \cap G_i \cap G_j = \emptyset$ for $1 \leqslant h < i < j \leqslant k$. If $F_1, \ldots, F_k$ is a system of closed sets satisfying conditions (i) of Lemma 2, then the components $K_1^\nabla, \ldots, K_n^\nabla$ of sets $F_1, \ldots, F_k$ satisfy the conditions

(1) $X = K_1^\nabla \cup \ldots \cup K_n^\nabla$, $\operatorname{diam}(K_i^\nabla) < \eta$, $\dim(K_i^\nabla \cap K_j^\nabla) \leqslant 0$, $K_h^\nabla \cap K_i^\nabla \cap K_j^\nabla = \emptyset$ for every system of distinct subscripts.

We can assume that there is an index i_0 such that

(2) $$K_i^\nabla \cap A \neq \emptyset \quad \text{if and only if} \quad i < i_0.$$

Consider the union $K_1^\nabla \cup \ldots \cup K_{i_0-1}^\nabla$ and let $K_1, \ldots, K_{j_0}$ be components of this union. Then the collection $K_1, \ldots, K_{j_0}, K_{i_0}^\nabla, \ldots, K_n^\nabla$ has all the required properties (i) of this Lemma by (1), (2) and the choice of the number η (these components are locally connected by Lemma 1).

Now we will prove

LEMMA 4. *If X is a 1-dimensional locally connected metric continuum, then there are a metric tree T and a continuous mapping f from T onto X such that* $\operatorname{card} f^{-1}(x) \leqslant 2$ *for each* $x \in X$ *and*

$$\dim\{x \in X: \operatorname{card} f^{-1}(x) \geqslant 2\} = \dim\{t \in T: \operatorname{card} f^{-1}(f(t)) \geqslant 2\} \leqslant 0.$$

Proof. We define, by induction, a sequence of 1-dimensional locally connected metric continua X_k and onto mappings $g_{j,k}: X_j \to X_k$ for $k < j$ such that

(1) $X_1 = X$,

(2) $\operatorname{card} g_{k+1,k}^{-1}(x) \leqslant 2$ for each $x \in X_k$ and $\dim A_k \leqslant 0$ where

$$A_k = \mathrm{cl}(\{x \in X_k: \operatorname{card} g_{k+1,k}^{-1}(x) = 2\}),$$

(3) if $\operatorname{card} g_{k+1,k}^{-1}(x) = 2$ for some $x \in X_k$ then $g_{1,k}^{-1} \circ g_{1,k}(x)$ is a one-point set,

(4) if S is a simple closed curve in X_k then

$$\operatorname{diam} g_{k,1}(S) < 1/(k-1).$$

Since X is a 1-dimensional locally connected metric continuum, we infer, by Lemma 3, that there is a finite system of locally connected continua $K_1, \ldots, K_{n_1}$ satisfying the conditions $X_1 = X = K_1 \cup \ldots \cup K_{n_1}$, $\operatorname{diam}(K_i) < 1$, $\dim(K_i \cap K_j) \leqslant 0$ for $i \neq j$ and $K_h \cap K_i \cap K_j = \emptyset$ for $h \neq i \neq j \neq h$. Since X_1 is a continuum we may assume that the sets K_i are indexed so that for each i there exists $x_i \in K_i \cap \bigcup_{j=1}^{i-1} K_j$. For each i there exists a continuum L_i homeomorphic to K_i and such that $L_i \cap \bigcup_{j=1}^{i-1} L_j = \{y_i\}$ where y_i corresponds to x_i in each of L_i and $\bigcup_{j=1}^{i-1} L_j$. Let $X_2 = L_1 \cup \ldots \cup L_{n_1}$. Let $g_{2,1}\colon X_2 \to X_1$ be the natural mapping. Then conditions (2), (3) and (4) are satisfied for $g_{2,1}$.

Let $\varepsilon_2 > 0$ be such that if $B_2 \subset X_2$ and $\operatorname{diam} B_2 < \varepsilon_2$ then $\operatorname{diam} g_{2,1}(B) < 1/2$. We write each L_i in X_2 as a finite union of locally connected continua $K_{i,1} \cup \ldots \cup K_{i,n_{1,i}}$ satisfying (i) of Lemma 3 with $\varepsilon = \varepsilon_2$ and $A = A_1$. Exactly as above we construct a continuum $X_3 = \bigcup \{L_{i,j}\colon i = 1, \ldots, n_1; j = 1, \ldots, n_{2,i}\}$ and an onto map $g_{3,2}\colon X_3 \to X_2$ satisfying (2), (3) and (4). Then $g_{3,2}^{-1}(L_i)$ is connected for each i.

Continuing this procedure with some obvious modifications we define $T = \operatorname{inv} \lim \{X_k, g_{j,k}\}$ where each X_k is a union of a finite family $L_{i_1,\ldots,i_{k-2},j}$ of continua and each $g_{j,k}^{-1}(K_{i_1,\ldots,i_{k-1},h})$ is a continuum. It follows that T is locally connected and T contains no simple closed curve. Hence, T is a dendrite. Also, $f = \operatorname{inv}\lim_{k \leqslant j} g_{j,k}$ is the required map.

LEMMA 5. *If $f\colon X \to Y$ is a continuous and finite-to-one mapping from a compact space onto a rim-finite (rational) continuum Y, then X is rim-finite (rational, respectively).*

Proof. Let x be an arbitrary point of X and let U be an open set in X containing x. Since $f^{-1}f(x)$ is finite, there are two open and disjoint sets G and H such that $f^{-1}f(x) \subset G \cup H$ and $x \in G \subset \operatorname{cl}(G) \subset U$.

Since $f(\operatorname{cl}(G) \setminus G)$ is a closed set which does not contain $f(x)$ we find an open set V such that $f(x) \in V \subset \operatorname{cl}(V) \subset Y \setminus f(\operatorname{cl}(G) \setminus G)$ and the boundary of V is finite (countable). Consider the set $W = G \cap f^{-1}(V)$. Then W is an open set containing x and contained in G. Moreover, the set W has a finite (countable) boundary.

COROLLARY 3. *If f is an open light mapping from a continuum X onto a metric rim-finite continuum Y, then there is a rim-finite continuum X_0 in X such that $f(X_0) = Y$.*

Indeed, since Y is locally connected (see [13], Lemma 2), there is a finite-to-one mapping g from a metric tree Z onto Y by Lemma 4. Theorem 1 implies the existence of a mapping h from Z into X such that $fh = g$. Then $f \mid h(Z)$ is finite-to-one; thus $x_0 = h(Z)$ is rim-finite by Lemma 5.

Similarly, we obtain

COROLLARY 4. *If f is an open light mapping from a metric continuum X onto a rational locally connected continuum Y, then there is a rational locally connected continuum X_0 in X such that $f(X_0) = Y$.*

4. Rim-type and arcs. If A is a subset of a continuum X we let A' denote the set of limit points of A. We let $A^{[0]} = A$. If X is an ordinal and $A^{[\lambda]}$ is defined for each $\lambda < \alpha$ we let

$$A^{[\alpha]} = \begin{cases} (A^{[\lambda]})' & \text{if} \quad \alpha = \lambda + 1, \\ \bigcap \{A^{[\lambda]} \mid \lambda < \alpha\} & \text{if} \quad \alpha \text{ is a limit ordinal.} \end{cases}$$

We say a space X has *rim-type* $\leqslant n$ if X has a basis U_i of open sets such that $(\mathrm{bd}(U_i))^{[n]} = \emptyset$ for each i. It is known that every rational metric continuum has rim-type being a countable ordinal (see [5], p. 290).

A continuum has rim-type 1 if and only if it is regular. It follows that a continuum of rim-type 1 is a tree if it is unicoherent. In [6], p. 64, Lelek gave an example of a continuum of rim-type 3 which is unicoherent and arcwise connected and which contains a non-arcwise connected subcontinuum. He asked in a letter whether there exists a unicoherent continuum X which has rim-type 2 and which contains a non-arcwise connected subcontinuum. It is the purpose of this section to answer Lelek's question in the affirmative by constructing such an example.

Example 2. We construct a sequence of disjoint continua X_i in Euclidean 3-space E^3. If P and Q are points of E^3 we let $\overline{PQ}$ denote the line segment with endpoints P and Q. We let X_{-1} be the following subcontinuum of the xy-plane in E^3 (see Figure 1).

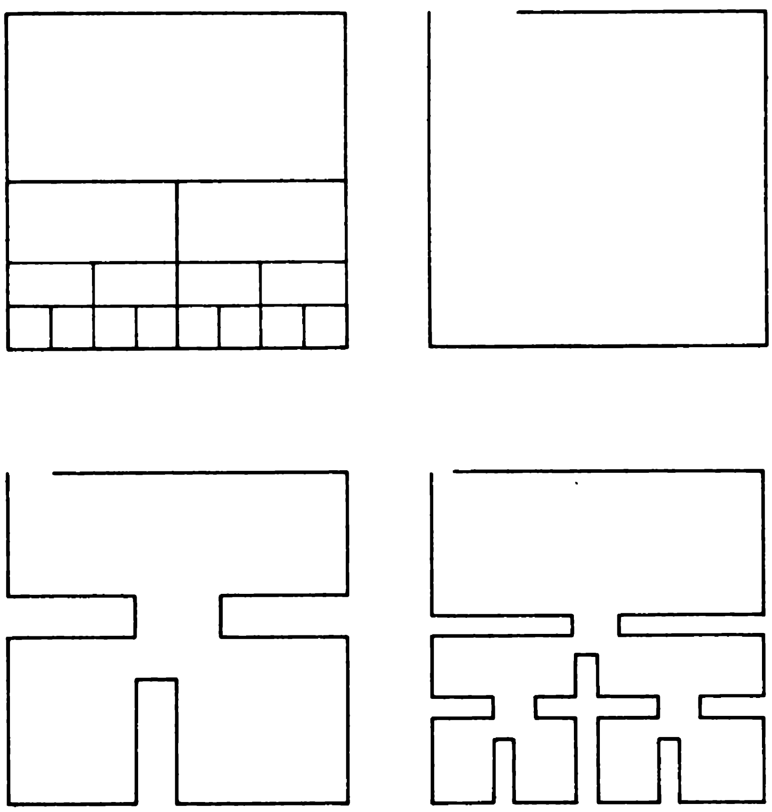

Fig. 1

$$X_{-1} = ([0, 1] \times \{0, 1, 2^{-1}, \ldots, 2^{-n}, \ldots\} \times \{0\}) \cup$$
$$\cup \{\overline{(p2^{-q}, 0, 0)(p2^{-q}, 2^{-q}, 0)} \mid q \in \{0, 1, \ldots\},\ p \in \{0, \ldots, 2^q\}\}.$$

We define a sequence $X_0, X_1, \ldots$ of pairwise disjoint polygonal arcs which limit to X_{-1} as follows:

Let X_0 be the polygonal arc in the plane $z = 1$ with consecutive vertices $(0, 1, 1)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(2^{-2}, 1, 1)$.

Suppose $n \geq 0$ and X_n is a polygonal arc in the plane $z = 2^{-n}$ with initial point $(0, 1, 2^{-n})$ and terminal point $(2^{-n-2}, 1, 2^{-n})$ at the consecutive vertices of X_n are prescribed. Let P and Q be two consecutive vertices in X_n where P separates Q and $(0, 1, 2^{-n})$ in X_n,

$$P = (p2^{-n} + \alpha 2^{-n-2}, r2^{-n} + \beta 2^{-n-2}, 2^{-n}),$$
$$Q = (q2^{-n} + \gamma 2^{-n-2}, t2^{-n} + \delta 2^{-n-2}, 2^{-n}),$$

where $p, q, r, t \in \{0, 1, \ldots, 2^n\}$ and $\alpha, \beta, \gamma, \delta \in \{-1, 0, 1\}$.

If $r, t \geq 1$ let

$$P' = (p2^{-n} + \alpha 2^{-n-3}, r2^{-n} + \beta 2^{-n-3}, 2^{-n-1}),$$
$$Q' = (q2^{-n} + \gamma 2^{-n-3}, t2^{-n} + \delta 2^{-n-3}, 2^{-n-1}).$$

In X_{n+1} let the segment $\overline{P'Q'}$ correspond to the segment $\overline{PQ}$ in X_n.

If $r = 1$ and $t = 0$ then $p = q$, $\alpha = \beta$, $\delta = 0$ and we let

$$\begin{aligned}
P' &= (p2^{-n} + \alpha 2^{-n-3}, 2^{-n} + \beta 2^{-n-3}, 2^{-n-1}),\\
R' &= (p2^{-n} + \alpha 2^{-n-3}, 2^{-n-1} + 2^{-n-3}, 2^{-n-1}),\\
S' &= ((2p+1)2^{-n-1} - 2^{-n-3}, 2^{-n-1} + 2^{-n-3}, 2^{-n-1}),\\
S'' &= ((2p+1)2^{-n-1} - 2^{-n-3}, 2^{-n-1}, 2^{-n-1}),\\
T' &= (p2^{-n} + \alpha 2^{-n-3}, 2^{-n-1}, 2^{-n-1}),\\
Q' &= (p2^{-n} + \alpha 2^{-n-3}, 0, 2^{-n-1}).
\end{aligned}$$

In X_{n+1} we let the polygonal arc with consecutive vertices P', R', S', S'', T', Q' correspond to the segment $\overline{PQ}$ in X_n.

If $r = t = 0$ then $q = p+1$, $\beta = \delta = 0$ and we let

$$\begin{aligned}
P' &= (p2^{-n} + \alpha 2^{-n-3}, 0, 2^{-n-1}),\\
R' &= ((2p+1)2^{-n-1} - 2^{-n-3}, 0, 2^{-n-1}),\\
S' &= ((2p+1)2^{-n-1} - 2^{-n-3}, 2^{-n-1} - 2^{-n-3}, 2^{-n-1}),\\
S'' &= ((2p+1)2^{-n-1}, 2^{-n-1} - 2^{-n-3}, 2^{-n-1}),\\
T' &= ((2p+1)2^{-n-1}, 0, 2^{-n-1}),\\
Q' &= (q2^{-n} + \gamma 2^{-n-3}, 0, 2^{-n-1}).
\end{aligned}$$

In X_{n+1} we let the polygonal arc with consecutive vertices P', R', S', S'', T', Q' correspond to the segment $\overline{PQ}$ in X_n.

Finally, if $r = 0$ and $t = 1$ then $p = q$, $\alpha = \gamma$, $\beta = 0$ and we let

$$\begin{aligned}
P' &= (p2^{-n}+\alpha 2^{-n-3}, 0, 2^{-n-1}),\\
R' &= (p2^{-n}+\alpha 2^{-n-3}, 2^{-n-1}, 2^{-n-1}),\\
S' &= ((2p-1)2^{-n-1}+2^{-n-3}, 2^{-n-1}, 2^{-n-1}),\\
S'' &= ((2p-1)2^{-n-1}+2^{-n-3}, 2^{-n-3}+2^{-n-1}, 2^{-n-1}),\\
T' &= (p2^{-n}+\alpha 2^{-n-3}, 2^{-n-1}+2^{-n-3}, 2^{-n-1}),\\
Q' &= (p2^{-n}+\alpha 2^{-n-3}, 2^{-n}+\delta 2^{-n-3}, 2^{-n-1}).
\end{aligned}$$

In X_{n+1} we let the polygonal arc with consecutive vertices P', R', S', S'', T', Q' correspond to the segment $\overline{PQ}$ in X_n.

By induction the polygonal arc X_n is defined for each non-negative integer n. Let X be the quotient space obtained from $X_{-1} \cup X_0 \cup X_1 \cup \ldots$ by identifying all of the points on the line $x = 0$, $y = 1$. Then X is clearly a continuum. It is easy to see that X has a basis of open sets U_i such that $\mathrm{bd}(U_i)$ has at most four limit points. Hence, X is a rational continuum of rim-type at most two. Since X is not locally connected, the rim-type of X is at least two. It follows from [6], Lemma 3, that X is unicoherent.

Remark. The above example may be modified so that X_{-1} contains no simple 4-od (i.e. no continuum homeomorphic to the following subcontinuum of the plane:

$$Y = ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 1]))$$

and X has a basis of open sets U_i such that $\mathrm{bd}(U_i)$ has at most three limit points. It is an unresolved question of Lelek whether the number three can be reduced to two or even one.

5. Compactifications of regular spaces. It is known (see [5], p. 290–291) that every regular metric space is topologically contained in a compact regular space and there is a space of order $\leqslant 4$ which is not topologically contained in any compact space of order $\leqslant 4$. We give a simple and stronger example.

Example 3. Let (r, φ) denote polar coordinates in the plane E^2. Put (compare Fig. 2)

$$\begin{aligned}
X = \{(r, \varphi)\colon 0 < r \leqslant 1, \varphi = 0 \text{ or } \pi\} &\cup \bigcup_{n=1}^{\infty} \{(1/n, \varphi)\colon 0 \leqslant \varphi \leqslant \pi\} \cup\\
&\cup \bigcup_{n=1}^{\infty} \{(r, \pi/2n)\colon 1/2k \leqslant r \leqslant 1/(2k-1), k = n, n+1, \ldots\} \cup\\
&\cup \bigcup_{n=1}^{\infty} \{(r, \pi/(2n+1))\colon 1/(2k+1) \leqslant r \leqslant 1/(2k+2), k = n, n+1, \ldots\}.
\end{aligned}$$

It is clear that the space X is of order $\leqslant 3$ at each of its points. Let X^* be a

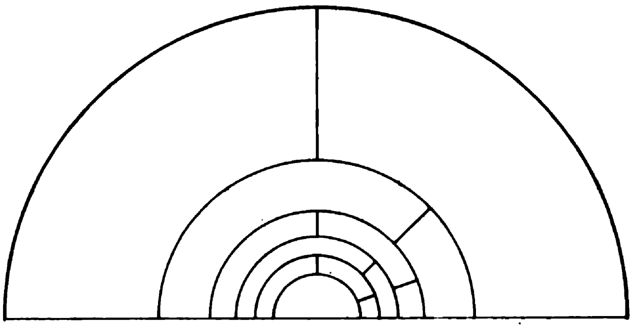

Fig. 2

compactification of X which is hereditarily locally connected. Since X is locally compact the growth $X^* - X$ is compact. Moreover, $X^* - X$ is a continuum because X is the union of an increasing sequence of continua which have connected complements in X.

Suppose that $a, b \in X^* - X$ and $a \neq b$. Then we find two sequences $\{a_n\}$ and $\{b_n\}$ such that $a = \lim a_n$, $b = \lim b_n$ and $a_n, b_n \in X$ for each $n = 1, 2, \ldots$ Since the one-point compactification $X \cup \{(0, 0)\}$ of X is locally connected at $(0, 0)$ and $\lim a_n = \lim b_n = (0, 0)$, we can find pairwise disjoint arcs $a_n b_n$ (we consider the subsequences of $\{a_n\}$ and $\{b_n\}$ if it is necessary) in X. We can assume that $\{a_n b_n\}$ is convergent in X^*. This implies that X^* has a non-degenerate convergence continuum, a contradiction. Therefore the unique hereditarily locally connected compactification of X is equal to $X \cup \{(0, 0)\}$.

Finally, remark that the point $\{(0, 0)\}$ in $X \cup \{(0, 0)\}$ is of order ω.

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