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ON EXTREME OPERATORS

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Introduction. The problem to characterize the extreme points of $U(E, F)$, the unit ball in the space of linear operators from a Banach space E to a Banach space F, was investigated by several authors (see, e.g., [1], [3], [4], [7], the literature cited there, and the survey article [2]). The most desirable characterization seems to be that an operator T is extreme iff its adjoint T^* maps a dense subset of the extreme points of the unit ball in F^* into the extreme points of the unit ball in E^*. (It is easy to see that an operator satisfying this condition is extreme.)

A theorem of that type, extending a result of Blumenthal et al. [1], is stated by Fakhoury in [4]. However, the proof given there appears to be incomplete (see Remark 2 below) and it is the aim of this note to provide a complete proof of that theorem.

Notation. For a compact Hausdorff space X we let $C(X)$ be the Banach space (under the supremum norm) of continuous real-valued functions on X and identify its dual with the space $M(X)$ of real Radon measures on X which will always carry the weak* (or vague) topology. If σ: $X \to X$ is an involution (i.e. σ composed with itself is the identity on X) which is continuous, we let

$$C_\sigma(X) := \{f \in C(X): f\circ\sigma = -f\}$$

be the Banach space of odd continuous functions on X. Its dual may be identified with the set

$$M_\sigma(X) := \{\mu \in M(X): \mu\circ\sigma = -\mu\}$$

of odd measures on X, where $\mu\circ\sigma(f) := \mu(f\circ\sigma)$ for f in $C(X)$. An identification of the fixed points of σ yields another compact Hausdorff space X' and another continuous involution σ': $X' \to X'$ such that $C_{\sigma'}(X')$ is isometric to $C_\sigma(X)$. Therefore, since we are only interested in the spaces $C_\sigma(X)$, we may and do always assume that σ has at most one fixed point.

Then the map

$$\omega\colon X \to M_\sigma^1(X) := \{\mu \in M_\sigma(X)\colon \|\mu\| \leqslant 1\}$$

defined by

$$x \to \omega_x := \tfrac{1}{2}(\varepsilon_x - \varepsilon_{\sigma(x)})$$

is a homeomorphic embedding (ε_x denotes the point mass concentrated at x). Further, the set $\mathrm{ex}M_\sigma^1(X)$ of extreme points in $M_\sigma^1(X)$ is exactly $\{\omega_x\colon x \in X, x \neq \sigma(x)\}$. Therefore, its closure is either $\mathrm{ex}M_\sigma^1(X) \cup \{0\}$ or $\mathrm{ex}M_\sigma^1(X)$ itself, depending on whether σ has a non-isolated fixed point or not.

The Theorem. We now state Fakhoury's theorem in a version slightly different from that given in [4].

THEOREM. *Let X and Y be compact Hausdorff spaces, let $\sigma\colon X \to X$ and $\tau\colon Y \to Y$ be continuous involutions, and suppose that X is a metrizable space. Let U be the convex set of linear operators from $C_\sigma(X)$ into $C_\tau(Y)$ with norm not greater than* 1.

Then for $T \in U$ the following statements are equivalent:

(1) *T is an extreme point of U.*

(2) *There exists a dense open set $D \subseteq \mathrm{ex}M_\tau^1(Y)$ such that $T^*(D) \subseteq \mathrm{ex}M_\sigma^1(X)$.*

We have already mentioned that (2) implies (1). To prove the converse we proceed in three steps, each step represented by a lemma.

The Lemmata. Our first lemma is a part of the proof of Theorem 1 in [1] and stated without proof.

LEMMA 1. *Let K be a compact Hausdorff space. Then the set-valued maps Φ^+ and Φ^- from the unit ball $M^1(K)$ of $M(K)$ into its non-empty compact convex subsets, defined by*

$$\Phi^+(\mu) := \{\nu \in M^1(K)\colon 0 \leqslant \nu \leqslant \mu^+\}, \qquad \Phi^-(\mu) := \{\nu \in M^1(K)\colon 0 \leqslant \nu \leqslant \mu^-\},$$

are lower semicontinuous in the sense of Michael [5].

LEMMA 2. *Let X and σ be as in the statement of the Theorem. Suppose that $\lambda \in M_\sigma^1(X)$ is such that the support of λ^+ contains at least two points. Then there exists a continuous map $g\colon M_\sigma^1(X) \to M_\sigma^1(X)$ such that*

(a) $g(-\mu) = -g(\mu)$ *for all* μ,

(b) $\|\mu \pm g(\mu)\| \leqslant 1$ *for all* μ,

(c) $g(\lambda) \neq 0$.

Proof. By our assumption there exists a compact subset F of X such that $0 < \lambda^+(F) < \lambda^+(X)$. Let $\nu \in M^1(X)$ be defined by $\nu(h) := \lambda^+(h\,|\,F)$ for h in $C(X)$. From Lemma 1 and from Michael's continuous selection theorem [6] we infer that there exist continuous maps f^+ and f^- from $M^1_\sigma(X)$ into $M^1(X)$ such that

$$0 \leqslant f^+(\mu) \leqslant \mu^+ \quad \text{and} \quad 0 \leqslant f^-(\mu) \leqslant \mu^- \quad \text{for all } \mu,$$

$$f^+(\lambda) = \nu, \quad f^-(\lambda) = \lambda^-, \quad f^+(-\lambda) = f^-(-\lambda) = 0.$$

Define $f\colon M^1_\sigma(X) \to M^1(X)$ by

$$f(\mu) := f^-(\mu)\|f^+(\mu)\| + f^+(\mu)\|f^-(\mu)\|.$$

Since the norm is weak* continuous on the cone of positive measures, f is continuous. Further, $\|\mu \pm f(\mu)\| \leqslant 1$ holds for all μ (see, e.g., [1], p. 751).

Finally, let

$$g(\mu) := \tfrac{1}{4}[f(\mu) - f(\mu)\circ\sigma + f(-\mu)\circ\sigma - f(-\mu)].$$

Then one easily verifies that g maps $M^1_\sigma(X)$ into itself, is continuous, and satisfies (a) and (b). It remains to show that $g(\lambda) \neq 0$ holds. If not, then

$$0 = \lambda^-\|\nu\| + \nu\|\lambda^-\| - (\lambda^-\circ\sigma)\|\nu\| - (\nu\circ\sigma)\|\lambda^-\|,$$

hence $\|\nu\|(\lambda^-\circ\sigma - \lambda^-) = \|\lambda^-\|(\nu - \nu\circ\sigma)$. Now $\lambda^-\circ\sigma = \lambda^+$ because λ is odd, and we get $\lambda = a(\nu - \nu\circ\sigma)$ for some $a > 0$. From $0 \leqslant \nu \leqslant \lambda^+$ we derive $0 \leqslant \nu\circ\sigma \leqslant \lambda^+\circ\sigma = \lambda^-$, which implies that ν and $\nu\circ\sigma$ are mutually singular. Therefore $\lambda^+ = a\nu$ must hold. But this fact and the choice of ν let us end up with the absurd inequality

$$0 < \lambda^+(X\setminus F) = a\nu(X\setminus F) = a\lambda^+\big((X\setminus F)\cap F\big) = 0.$$

The next lemma is modelled after a similar lemma due to Sharir [7], Theorem 1.

LEMMA 3. *Let Y and τ be as in the statement of the Theorem, E a Banach space, and $S\colon E \to C_\tau(Y)$ an extreme point of $U\big(E, C_\tau(Y)\big)$. Then the set*

$$D := \{\mu \in \operatorname{ex} M^1_\tau(Y)\colon \|S^*(\mu)\| = 1\}$$

is dense in $\operatorname{ex} M^1_\tau(Y)$.

Proof. We first observe that $\operatorname{ex} M^1_\tau(Y)$ is open in its closure, and therefore a Baire space. The map $e^* \to \|e^*\|$ is weak* lower semicontinuous on E^*, which implies that the set

$$D_n := \{\mu \in \operatorname{ex} M^1_\tau(Y)\colon \|S^*(\mu)\| > 1 - 1/n\}$$

is open for each positive integer n. Hence, by Baire's category theorem, the lemma is proved if we can show that each of the sets D_n is dense.

Suppose this is false for some n. Choose μ_0 in $\mathrm{ex}M_\tau^1(Y)$ not in the closure of D_n. Then $\mu_0 \neq 0$ and $\mu_0 \neq -\mu_0$; hence there exists a continuous real-valued function k on the closure of $\mathrm{ex}M_\tau^1(Y)$ such that $0 \leqslant k \leqslant 1$, $k(\mu_0) = 1$, $k(-\mu_0) = 0$, and $k(\mu) = 0$ for all μ in D_n. Then the function h on Y defined by

$$h(y) := \tfrac{1}{2}[k(\omega_y) - k(-\omega_y)]$$

is in $C_\tau(Y)$ and $0 < \|h\| \leqslant 1$. Choose e^* in E^* with $\|e^*\| = 1$ and define $H\colon E \to C_\tau(Y)$ by

$$[H(e)](y) := (1/n)h(y)e^*(e)$$

for e in E and y in Y. Then H is linear, $\|H\| \leqslant 1$, $H \neq 0$, and it is easily checked that $\|S \pm H\| \leqslant 1$ holds. Consequently, S would not be extreme and we obtain a contradiction.

Proof of the Theorem. Let T be an element of $\mathrm{ex}\,U$. We first show that $T^*(\mathrm{ex}M_\tau^1(Y))$ is contained in $Z := \{a\omega_x\colon 0 \leqslant a \leqslant 1, x \in X\}$. Suppose that $T^*(\omega_a) =: \lambda$ is not in Z for some a in Y with $a \neq \tau(a)$. Then the support of λ^+ contains more than one point. Let $g\colon M_\sigma^1(X) \to M_\sigma^1(X)$ be a map as described in Lemma 2 and define a linear operator $G\colon C_\sigma(X) \to C_\tau(Y)$ by

$$[G(h)](y) := \langle g(T^*(\omega_y)), h\rangle$$

for y in Y and h in $C_\sigma(X)$. Then $G \neq 0$ and $\|T \pm G\| \leqslant 1$, which is impossible because T is an extreme point of U.

Next, we infer from Lemma 3 that $\|T^*(\nu)\| = 1$ for all ν in a dense subset D of $\mathrm{ex}M_\tau^1(Y)$. Therefore, the image $T^*(D)$ is contained in $\{\omega_x\colon x \in X, x \neq \sigma(x)\}$ which is $\mathrm{ex}M_\sigma^1(X)$. We infer also that $T^*(\mathrm{ex}M_\tau^1(Y))$ is contained in the closure of $\mathrm{ex}M_\sigma^1(X)$ which in turn is contained in $\mathrm{ex}M_\sigma^1(X) \cup \{0\}$. Consequently, D is in fact equal to $\mathrm{ex}M_\tau^1(Y) \setminus [(T^*)^{-1}(0)]$, and therefore open in $\mathrm{ex}M_\tau^1(Y)$.

This completes the proof of the Theorem.

Remarks. 1. Denoting, as we already did, the homeomorphic embeddings of X and Y into $\mathrm{ex}M_\sigma^1(X)$ and $M_\tau^1(Y)$ by the same symbol ω and defining $\varphi\colon Y \to X$ by $\varphi(y) := \omega^{-1}T^*\omega_y$, we see that, under the hypotheses of the Theorem, an operator T in $\mathrm{ex}\,U$ is induced in the following sense:

There exists a continuous map $\varphi\colon Y \to X$ such that

$\varphi(\tau(y)) = \sigma(\varphi(y))$ for all $y \in Y$,

$[T(f)](y) = f(\varphi(y))$ for all $y \in Y$ and $f \in C_\sigma(X)$,

$\varphi^{-1}(x_0) \cap (Y \setminus \{y_0\})$ has an empty interior, where x_0 and y_0 are the fixed points — if there are any — of σ and τ, respectively.

Since, conversely, an induced operator is always extreme in U and since $C_\sigma(X)$ is separable iff X is metrizable, we also get Fakhoury's theorem as in [4].

2. The gap in the proof of Theorem 14 in [4] occurs on page 20^{17}, where it is implicitly assumed that for a non-zero non-extreme element ν of $M_\sigma^1(X)$ the set

$$\{\nu' \in M_\sigma^1(X) : \nu' = f\nu,\ f \in L^1(\nu),\ 1 \leqslant f \leqslant 2\}$$

contains elements different from ν. But this can be true only if $\|\nu\| < 1$. This means that from the arguments in [4] one can only conclude that for every μ in $\mathrm{ex} M_\tau^1(Y)$ the norm of $T^*(\mu)$ is either 0 or 1. Obviously, this fact combined with Lemma 2 would be sufficient to prove the Theorem; however, since Lemma 3 does not depend on any special property of E, it seemed justified to include it and use it in lieu of Fakhoury's result.

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