COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 2

A GENERALIZATION OF STEINHAUS' THEOREM TO COORDINATEWISE MEASURE PRESERVING BINARY TRANSFORMATIONS

BY

MARCIN E. KUCZMA (WARSZAWA)

The aim of this paper is to prove the following theorem:

THEOREM. *Let X be a topological space, let $\mathscr{B}(X)$ denote the σ-field of Borel subsets of X, and let m be a σ-finite regular measure on $\mathscr{B}(X)$. Further, let $f\colon X\times X\to X$ be a binary transformation with the following properties:*

1° *f is continuous and $\big(\mathscr{B}(X)\times\mathscr{B}(X),\ \mathscr{B}(X)\big)$-measurable; the equation $f(x,y)=z$ has a unique solution y, respectively x, for any fixed values of x, z, respectively y, z; this solution depends continuously and $\big(\mathscr{B}(X)\times\mathscr{B}(X),\ \mathscr{B}(X)\big)$-measurably on x, z, respectively on y, z.*

2° *For fixed $x, y\in X$ the transformations $f(x,\cdot)$ and $f(\cdot,y)$ preserve measure.*

Then for any two sets $A, B\in\mathscr{B}(X)$ of positive measure the set $f(A\times B)$ has interior points.

If $X=R$, the real line with the usual topology, m is the Lebesgue measure and $f(x,y)=x+y$, then the theorem asserts that the set $A+B=\{x+y\colon x\in A,\ y\in B\}$ contains an interval. This is a classical result due to Steinhaus [5].

The case of $X=R^n$, with the n-dimensional Lebesgue measure and $f(x,y)=x+y$, which is but a slight generalization of Steinhaus' result, may also be regarded as classical. Various proofs have been supplied by several authors. Certain modifications of this result have also been proved. For a more complete bibliography of the subject the reader is referred to [3] and [4].

Steinhaus' theorem for $X=R^n$ is a particular case of that in which X is a locally compact group, m is the Haar measure (left or right invariant), and f is the group operation. The assertion in this case follows from the fact that the convolution of the characteristic functions of sets of finite measure is a continuous function (cf., e.g., [2], Theorem (20.16), p. 296). This observation is due to A. Weil (see [2], Corollary (20.17)).

The theorem presented in this paper comprises Weil's result in the case of a σ-compact unimodular group (i.e., a group in which the left invariant measure is also right invariant). It also shows that in this case not all algebraic properties of the group operation are essential.

1. We start with some definitions and notation. If X is any set and if f is a binary transformation $X \times X \to X$, then the mapping $X \to X$ obtained from f by fixing one variable is denoted by $f(x, \cdot)$ or by $f(\cdot, y)$ (i.e., the first of these symbols stands for the mapping $y \mapsto f(x, y)$ and the second one for $x \mapsto f(x, y)$). If $A, B \subset X$, $x, y \in X$, then $f(A, y)$, respectively $f(x, B)$, is the image of the set A, respectively B, under the mapping $f(\cdot, y)$, respectively $f(x, \cdot)$; this notation is used for brevity instead of $f(A \times \{y\})$, respectively $f(\{x\} \times B)$.

A one-to-one mapping of X onto X is called a *bijection*.

A *measurable space* is a pair $(X, \mathscr{M})$, where $\mathscr{M}$ is a σ-field of subsets of X; a *measure space* is a triple $(X, \mathscr{M}, m)$ such that $(X, \mathscr{M})$ is a measurable space and m is a non-negative measure defined on $\mathscr{M}$. A bijection $g: X \to X$ is called *bimeasurable* if both g and g^{-1} are measurable. A measurable mapping $g: X \to X$ is said to *preserve measure* if $m\big(g^{-1}(E)\big) = m(E)$ for all $E \in \mathscr{M}$. In the case of a bimeasurable g this is clearly equivalent to: $m\big(g(E)\big) = m(E)$ for all $E \in \mathscr{M}$. Sets of measure zero are called *nullsets*. A measurable mapping $g: X \to X$ is called *non--singular* if $g^{-1}(E)$ is a nullset for any nullset $E \subset X$.

Throughout the sequel we restrict attention to σ-finite measures. For a σ-finite measure m we denote by M the *product measure* $m \times m$, i.e., the measure defined on the σ-field $\mathscr{M} \times \mathscr{M}$ of subsets of $X \times X$ generated by the sets $A \times B$ with $A, B \in \mathscr{M}$, determined uniquely by the condition

$$M(A \times B) = m(A)m(B) \quad \text{for } A, B \in \mathscr{M}.$$

If X is a topological space, we denote by $\mathscr{B}(X)$ the *σ-field of Borel subsets* of X, i.e., the σ-field generated by the open subsets of X. A measure m defined on $\mathscr{B}(X)$ is *regular* if it is finite on compact sets and satisfies

$$m(E) = \sup\{m(K): K \subset E,\ K \text{ compact}\} = \inf\{m(U): U \supset E,\ U \text{ open}\} \tag{1}$$

for any set $E \in \mathscr{B}(X)$. Thus, a regular measure is fully determined by its values on all open sets or on all compact sets.

Our terminology is adopted from [1].

Let X be any set and let $f: X \times X \to X$. We introduce the following solvability condition:

(S) The equation $f(x, y) = z$ has a unique solution y, respectively x, for any fixed values of x, z, respectively y, z; this solution will be denoted by $\varphi(x, z)$, respectively $\psi(z, y)$.

Condition (S) means that, for fixed x, y, $z \in X$, the mappings

$$f(\cdot, y),\ f(x, \cdot), \tag{2}$$

hence also

$$\varphi(\cdot, z),\ \psi(z, \cdot), \tag{3}$$

are bijections. The relations

$$y = \varphi(x, z), \quad x = \psi(z, y) \quad \text{and} \quad z = f(x, y)$$

are equivalent. Clearly, for a fixed $z \in X$, mappings (3) are mutually inverse.

We first prove

PROPOSITION 1. *Let X be a topological space and let m be a σ-finite regular measure on $\mathscr{B}(X)$. If $f\colon X \times X \to X$ satisfies condition* (S), *ψ is continuous, $f(x, \cdot)$ and $\psi(z, \cdot)$ are measurable and preserve measure (for any fixed x, $z \in X$) and if φ is $(\mathscr{B}(X) \times \mathscr{B}(X), \mathscr{B}(X))$-measurable, then for any two sets A, $B \in \mathscr{B}(X)$ of positive measure the set $f(A \times B)$ has interior points.*

Remark 1. Clearly, $\varphi(x, z)$ and $\psi(z, y)$ are analogues of the operations $x^{-1}z$ and zy^{-1} in the group case. The proof given below is nothing else but an adaptation of Weil's proof of Steinhaus' theorem. The function ω introduced below (formula (6)) is a direct analogue of the convolution $\chi_A * \chi_B$ with respect to the left Haar measure (cf. formula (10)).

Proof of Proposition 1. Let A, $B \in \mathscr{B}(X)$ and suppose that

$$m(A) > 0 \quad \text{and} \quad m(B) > 0. \tag{4}$$

We have to show that

$$\operatorname{int}(f(A \times B)) \neq \emptyset. \tag{5}$$

According to (1), A and B contain compact sets of positive measure; we may thus assume that they are themselves compact.

Consider the function $\omega\colon X \to R$ given by

$$\omega(z) = m(A \cap \psi(z, B)). \tag{6}$$

We shall show that ω is continuous and satisfies

$$\int_X \omega(z)\, dm(z) = m(A)m(B). \tag{7}$$

Fix $z_0 \in X$ and $\varepsilon > 0$. The set $\psi(z_0, B)$ is compact, hence of finite measure. In view of (1) there exists an open set $U \supset \psi(z_0, B)$ such that

$$m(U \setminus \psi(z_0, B)) < \varepsilon. \tag{8}$$

By assumption, ψ is continuous; consequently, B being compact, it is easy to find a neighbourhood W of z_0 such that $\psi(W\times B)\subset U$, i.e.,

$$\psi(z, B)\subset U \quad \text{for } z\in W. \tag{9}$$

Further, for all $z\in X$ we have

$$m(\psi(z, B)) = m(B) = m(\psi(z_0, B)),$$

since $\psi(z,\cdot)$ preserves measure. Hence, by (8) and (9),

$$m(U\setminus\psi(z, B)) < \varepsilon \quad \text{for } z\in W.$$

Writing, for brevity, $D = \psi(z, B)$ and $D_0 = \psi(z_0, B)$ we thus obtain

$$\begin{aligned}|\omega(z)-\omega(z_0)| &= |m(A\cap D)-m(A\cap D_0)| = |m(A\cap D\setminus D_0)-m(A\cap D_0\setminus D)|\\ &\leqslant m(D\setminus D_0)+m(D_0\setminus D)\leqslant m(U\setminus D_0)+m(U\setminus D) < 2\varepsilon \quad \text{for } z\in W,\end{aligned}$$

and so ω is continuous.

Now we prove equality (7). Observe that

$$\omega(z) = \int_X \chi_A(x)\chi_B(\varphi(x, z))\,dm(x); \tag{10}$$

in fact, the integrand equals 1 if and only if $x\in A$ and $\varphi(x, z)\in B$; equivalently, iff $x\in A$ and $x\in\psi(z, B)$ (and is zero otherwise). Thus the right-hand sides of (6) and (10) are equal. Integrating (10) over X with respect to z and applying the Fubini theorem, we obtain

$$\int_X \omega(z)\,dm(z) = \int_X \chi_A(x)\Big[\int_X \chi_B(\varphi(x, z))\,dm(z)\Big]dm(x).$$

But $\varphi(x, z)\in B$ if and only if $z\in f(x, B)$. Hence the integral in square brackets is equal to $m(f(x, B))$, i.e., to $m(B)$, since, by assumption, $f(x,\cdot)$ preserves measure. This proves (7).

Consider the set $\Omega = \{z\in X\colon \omega(z) > 0\}$; this set is open by the continuity of ω and is non-empty in view of (4) and (7). For $z\in\Omega$ by (6) we have $A\cap\psi(z, B)\neq\emptyset$, i.e., $z\in f(A\times B)$. Consequently, $\Omega\subset f(A\times B)$ and assertion (5) follows. The proof is complete.

2. The assumption that $\psi(z,\cdot)$ preserves measure is rather inconvenient and it is natural to ask whether it can be eliminated, e.g., by requiring that f preserves measure in both variables (so that measure invariance assumptions could be expressed in terms of f only). In what follows we show that this can be done under the conditions of the Theorem formulated at the outset.

For convenience we introduce two further conditions, (CS) and (MS), which are extensions of condition (S) concerning a mapping $f\colon X\times X\to X$. Condition (S) makes sense if X is any set, (CS) and (MS) are meaningful if X is a topological space and a measurable space, respectively:

(CS) f satisfies (S), and f, φ and ψ are continuous.

(MS) f satisfies (S), and f, φ and ψ are $(\mathscr{M}\times\mathscr{M}, \mathscr{M})$-measurable. (Clearly, φ and ψ are the mappings defined in (S); in condition (MS), $\mathscr{M}$ denotes the σ-field of measurable subsets of X.)

Condition (CS) implies that mappings (2) and (3) are homeomorphisms. Similarly, condition (MS) implies that they are bimeasurable.

If X is a topological space, then condition (CS) makes sense as well as condition (MS) with $\mathscr{M} = \mathscr{B}(X)$. Let us denote this latter condition by (BS):

(BS) f satisfies condition (MS) with respect to the σ-field $\mathscr{M} = \mathscr{B}(X)$.

Condition (CS) implies (BS) in the case of a topological space with a countable base, since then $\mathscr{B}(X\times X) = \mathscr{B}(X)\times\mathscr{B}(X)$.

Note that conditions (CS) and (BS) are jointly equivalent to assumption 1° of the Theorem.

The following lemma gives an alternative form of conditions (S), CS), and (MS).

LEMMA 1. *Let F and G be transformations $X\times X\to X\times X$ defined by*

$$F(x,y) = \big(x, f(x,y)\big) \quad \text{and} \quad G(x,y) = \big(f(x,y), y\big). \tag{11}$$

Then conditions (S), (CS), *and* (MS) *are accordingly equivalent to the requirement that the transformations F and G are bijections, homeomorphisms, and $(\mathscr{M}\times\mathscr{M}, \mathscr{M}\times\mathscr{M})$-bimeasurable mappings.*

Proof. The assertion concerning condition (S) is obvious. Evidently, f is continuous, respectively $(\mathscr{M}\times\mathscr{M}, \mathscr{M})$-measurable, if and only if F (and G) is continuous, respectively $(\mathscr{M}\times\mathscr{M}, \mathscr{M}\times\mathscr{M})$-measurable. Since

$$F^{-1}(x,z) = \big(x, \varphi(x,z)\big) \quad \text{and} \quad G^{-1}(z,y) = \big(\psi(z,y), y\big),$$

F^{-1} and G^{-1} are continuous, respectively measurable, if and only if so are φ and ψ.

The next lemma shows how the fact that f coordinatewise preserves measure can be expressed in terms of mappings (11).

LEMMA 2 ([1]). *Let $(X, \mathscr{M}, m)$ be a measure space with m σ-finite. Let $f\colon X\times X\to X$ be a transformation satisfying condition* (MS) *and let $F\colon X\times X\to X\times X$ be defined by* (11). *Then the following two conditions are equivalent:*

(i) *F preserves the product measure $M = m\times m$.*

(ii) *For every $B\in\mathscr{M}$ and for almost every $x\in X$ we have*

$$m\big(f(x,B)\big) = m(B).$$

([1]) This lemma has been communicated to the author by M. Misiurewicz.

Remark 2. In condition (ii) "for almost every $x \in X$" means "for all $x \in X$ off a nullset, which depends on B". If we additionally assume that the measure m is uniquely determined by its values on the members of some countable subfamily $\mathcal{M}_0 \subset \mathcal{M}$ (i.e., if there exists no measure $m' \neq m$ such that $m'(E) = m(E)$ for $E \in \mathcal{M}_0$), then this nullset can be chosen independently of B. Consequently, in that case (i) and (ii) are equivalent to

(iii) *For almost every $x \in X$ the mapping $f(x, \cdot)$ preserves measure.*

Proof of Lemma 2. F is bimeasurable by Lemma 1. For $A, B \in \mathcal{M}$ we have

$$F(A \times B) = \{(x, z)\colon\ x \in A,\ z \in f(x, B)\}.$$

Hence

$$M\big(F(A \times B)\big) = \int_A m\big(f(x, B)\big)\, dm(x). \tag{12}$$

By the definition of the product measure, we also have

$$M(A \times B) = m(A)m(B) = \int_A m(B)\, dm(x). \tag{13}$$

Now, condition (ii) means precisely that, given any set $B \in \mathcal{M}$, the integrand in (12) coincides a.e. with the integrand in (13), and this is the case if and only if integrals (12) and (13) are equal for all $A \in \mathcal{M}$. Consequently, condition (ii) is equivalent to the equality of the right-hand sides of (12) and (13) for all $A, B \in \mathcal{M}$. Condition (i) is just the equality of the left-hand sides of (12) and (13) for all $A, B \in \mathcal{M}$. Thus (i) and (ii) are equivalent, as asserted.

From Lemma 2 we easily derive

LEMMA 3. *Let $(X, \mathcal{M}, m)$ and f be as in Lemma 2 and suppose that for almost every x and y the transformations $f(x, \cdot)$ and $f(\cdot, y)$ preserve measure. Then for every $E \in \mathcal{M}$ and for almost every $z \in X$ we have*

$$m\big(\varphi(E, z)\big) = m(E) = m\big(\psi(z, E)\big).$$

Proof. Consider the bimeasurable transformations $F, G\colon X \times X \to X \times X$ defined by (11). In virtue of Lemma 2, F preserves the measure $M = m \times m$, and so does G, by symmetry. Hence, also the mapping $\Phi = G \circ F^{-1}$ is bimeasurable and preserves measure. Further, observe that $\Phi(x, z) = \big(z, \varphi(x, z)\big)$. Applying once more Lemma 2 with f replaced by φ' and F by Φ', where $\varphi'(z, x) = \varphi(x, z)$ and $\Phi'(z, x) = \Phi(x, z)$, we conclude that φ has the asserted property. The assertion concerning ψ follows by symmetry.

Remark 3. If m has the property formulated in Remark 2, then the assumptions of Lemma 3 imply (cf. Remark 2):

(iv) *For almost every $z \in X$ the mappings $\varphi(\cdot, z)$ and $\psi(z, \cdot)$ preserve measure.*

This is a partial answer to the question whether the invariance of measure with respect to mappings (2) implies its invariance with respect to mappings (3). What we actually need, however, is an analogue of (iv) holding for all, not only for almost all $z \in X$. Observe that Lemmas 2 and 3 are of purely measure-theoretic nature, and thus nothing better than (iv) can be expected. In order to obtain the invariance with respect to mappings (3) for all z we have to pass to the case where X is a topological space.

First, we show that under the assumptions of the Theorem a property which holds a.e. is automatically satisfied on a dense set.

LEMMA 4. *Let X be a topological space, let m be a non-zero regular measure on $\mathscr{B}(X)$, and let $f\colon X \times X \to X$ be a mapping such that, for every fixed $x, y \in X$, $f(x, \cdot)$ is a surjection and $f(\cdot, y)$ is continuous and non-singular. Then every non-empty open set has positive measure.*

Proof. Suppose that $m(W) = 0$ for some non-empty open set W. Take an $x \in X$. Since $f(x, \cdot)$ is surjective, there exists a $y \in X$ such that $f(x, y) \in W$. Fix such a y and write $g = f(\cdot, y)$. By assumption, g is continuous and non-singular. Hence the set $U = g^{-1}(W)$ is open and $m(U) = 0$. Clearly, $x \in U$. It follows that every point $x \in X$ has a neighbourhood of measure zero; thus all compact sets have measure zero, and so, by (1), m is the zero measure, a contradiction.

Now we can prove the invariance of m with respect to mappings (3).

PROPOSITION 2. *Under the assumptions of the Theorem the mappings $\varphi(\cdot, z)$ and $\psi(z, \cdot)$ preserve measure for every $z \in X$.*

Remark 4. This proposition corresponds to the fact that in a unimodular group not only translations, but also symmetries of the form $x \mapsto x^{-1}z$ and $y \mapsto zy^{-1}$ (z fixed) preserve the Haar measure.

Proof of Proposition 2. Fix $z_0 \in X$, $\varepsilon > 0$ and a compact set $B \subset X$. The set $\psi(z_0, B)$ is compact, hence of finite measure. By the regularity of m (formula (1)), there is an open set $U \supset \psi(z_0, B)$ such that

$$m(U) - m\big(\psi(z_0, B)\big) < \varepsilon. \tag{14}$$

By assumption, ψ is continuous; consequently, B being compact, it is easy to find a neighbourhood W of z_0 such that $\psi(W \times B) \subset U$, i.e.,

$$\psi(z, B) \subset U \quad \text{for } z \in W. \tag{15}$$

In view of Lemma 3 (with $\mathscr{M} = \mathscr{B}(X)$), we have

$$m\big(\psi(z, B)\big) = m(B) \tag{16}$$

for $z \in X \setminus Z$, where Z is a nullset (depending on B). The set $X \setminus Z$ is dense by Lemma 4 (unless $m \equiv 0$, but in this case the proposition is obvious), and so it intersects W. Choose any point $z \in W \setminus Z$. This point satisfies conditions (15) and (16), which imply that $m(B) \leqslant m(U)$, and this together with (14) gives

$$m\big(\psi(z_0, B)\big) > m(B) - \varepsilon.$$

Assuming that z_0 and B are fixed and passing with ε to 0, we obtain

$$m\big(\psi(z_0, B)\big) \geqslant m(B), \tag{17}$$

which holds for any $z_0 \in X$ and any compact set $B \subset X$.

In view of the symmetry of assumptions with respect to the factors of the product $X \times X$ ("x-axis" and "y-axis"), we have also

$$m\big(\varphi(A, z_0)\big) \geqslant m(A) \tag{18}$$

for any $z_0 \in X$ and any compact set $A \subset X$. Setting in (17) and (18) $B = \varphi(A, z_0)$, i.e., $A = \psi(z_0, B)$, we infer that (17) and (18) are actually equalities. This means that

$$m\big(\varphi(E, z_0)\big) = m(E) = m\big(\psi(z_0, E)\big) \tag{19}$$

for $E \subset X$ compact. Since $\varphi(\cdot, z_0)$ and $\psi(z_0, \cdot)$ are homeomorphisms, the measures

$$\mathscr{B}(X) \ni E \mapsto m\big(\varphi(E, z_0)\big) \quad \text{and} \quad \mathscr{B}(X) \ni E \mapsto m\big(\psi(z_0, E)\big)$$

are regular. Consequently, (19) holds for all sets $E \in \mathscr{B}(X)$. This completes the proof, since $z_0 \in X$ was chosen arbitrarily.

Now the Theorem follows immediately from Propositions 1 and 2.

REFERENCES

[1] P. R. Halmos, *Measure theory*, New York 1950.

[2] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I, Berlin 1963.

[3] M. E. Kuczma and M. Kuczma, *An elementary proof and an extension of a theorem of Steinhaus*, Glasnik Matematički 6 (26) (1971), p. 11-18.

[4] E. Marczewski, *On translations of sets and a theorem of Steinhaus* [in Polish], Prace Matematyczne 1 (2) (1955), p. 256-263.

[5] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fundamenta Mathematicae 1 (1920), p. 93-104.

Reçu par la Rédaction le 3. 4. 1974;
en version modifiée le 12. 4. 1975