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PARAMETRIC DIFFERENTIATION

BY

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1. Introduction. By a *parameter* φ we mean a continuous real-valued function defined on $[0,1]$ with $\varphi(0) = 0$ and having the property that there is a positive number δ_φ such that both of the functions φ and ψ $(\psi(h) \equiv \varphi(h)+h)$ are strictly monotone on $[0, \delta_\varphi)$. Let P be the collection of all parameters. If f is a real-valued function defined on $\boldsymbol{R}$, $x \in \boldsymbol{R}$, and $\varphi \in P$, define the *lower φ-derivative* of f at x by

$$D_\varphi f(x) = \liminf_{h\to 0^+} \big[f\big(x-\varphi(h)\big) - f\big(x-\varphi(h)-h\big)\big]/h.$$

The *upper φ-derivative*, $D^\varphi f(x)$, is defined analogously; and if $D_\varphi f(x) = D^\varphi f(x)$, denote the common value by $f^\varphi(x)$ and call this the *φ-derivative* of f at x. This notion was introduced by Sindalovskiĭ [6]. He, however, did not require φ and ψ to be monotone and continuous.

The parameter class P naturally splits into three sets, S, R, and L, according to whether for $0 < h < \delta_\varphi$ we have $-h < \varphi(h) < 0$, $\varphi(h) < -h$, or $\varphi(h) > 0$, respectively. From the viewpoint of a generalized type of differentiation we found R and L to be not very interesting, since, for example, it is an elementary matter to see that if $\varphi \in R$ $[L]$ and a continuous function f has a φ-derivative at x, then f has a right [left] derivative at x. Consequently, most of the results in this paper concern parameters φ in S.

Following Sindalovskiĭ, we say that a parameter φ has *property S_d'* if, for every set P which has zero as a point of density, either $\varphi^{-1}(P)$ or $\psi^{-1}(P)$ has positive lower density at zero; and φ has *property S_d''* if there are a ϱ $(0 < \varrho < 1)$ and a δ $(0 < \delta < \delta_\varphi)$ such that for every $x \in (0, \delta)$ and every closed set $P \subseteq [0, x]$ with $|P| > (1-\varrho)x$ we have either

$$|\varphi(P)| > \varrho\,|\varphi(x)| \quad \text{and} \quad |\varphi(x)| > \varrho x$$

or

$$|\psi(P)| > \varrho\,|\psi(x)| \quad \text{and} \quad |\psi(x)| > \varrho x,$$

where ϱ depends only on φ. We then let S^* denote the class of all parameters in S which have properties S'_d and S''_d. Examples of such parameters are those of the form $\varphi(h) = ch^a$, where $c < 0$, $a \geqslant 1$, and $c \neq -1$ if $a = 1$. In particular, the familiar notion of symmetric differentiation is obtained by taking $c = -1/2$ and $a = 1$.

The following result is a special case of a more general theorem in [6]:

THEOREM S. *Let* $f\colon \boldsymbol{R} \to \boldsymbol{R}$ *be measurable and let* $\varphi \in S^*$. *If* $D^\varphi f(x) < +\infty$ *for almost every* $x \in \boldsymbol{R}$, *then* f *is differentiable almost everywhere.*

In particular, this theorem entails that if f is φ-differentiable almost everywhere, then it is differentiable almost everywhere. In Section 2 of this paper we further show that f must be differentiable except on a first category set. (If a property holds for every x except for those in a first category set, we adopt the notation of saying that the property holds for *nearly every* x.)

In Section 3 we prove a monotonicity theorem involving the φ-derivative and examine some consequences. One consequence is that if a measurable function f has a finite φ-derivative on $\boldsymbol{R}$, where $\varphi \in S^*$, and if f^φ is bounded from above or from below, then f^φ must belong to the Baire class one.

2. Parametric and ordinary differentiation. From Theorem S it follows that if $\varphi \in S^*$ and a function f is φ-differentiable almost everywhere on $\boldsymbol{R}$, then f is actually differentiable almost everywhere on $\boldsymbol{R}$. In this section we augment this result by showing that such an f will be differentiable nearly everywhere. We begin by establishing a relation between the φ-derivatives and the so-called strong derivatives of a function. The *lower strong derivative* of f at x is defined by

$$D_* f(x) = \liminf_{\substack{(\xi,\eta)\to(x,x)\\ \xi\neq\eta}} [f(\xi)-f(\eta)]/(\xi-\eta),$$

and the *upper strong derivative* is defined analogously (cf. [1]).

THEOREM 1. *If* $f\colon \boldsymbol{R} \to \boldsymbol{R}$ *has the Baire property, then for any* $\varphi \in P$ *both of the following equalities hold nearly everywhere*:

(i) $D_\varphi f(x) = D_* f(x)$,

(ii) $D^\varphi f(x) = D^* f(x)$.

Proof. Sentence (ii) follows from (i) by considering the function $-f$; consequently, we need only to prove (i).

Clearly, $D_\varphi f(x) \geqslant D_* f(x)$ for every $x \in \boldsymbol{R}$, and hence it suffices to show that the set $A(f) = \{x\colon D_\varphi f(x) > D_* f(x)\}$ is of the first category. Furthermore, $A(f)$ is the countable union of all the sets

$$A(f, a) \equiv \{x\colon D_\varphi f(x) > a > D_* f(x)\} \quad (a \text{ rational}),$$

and as for each $A(f, a)$ we have $A(f, a) = A(g_a, 0)$, where $g_a(x) = f(x) - ax$, we need only to show that $A(f, 0)$ is of the first category. For each positive integer n, set

$$\Phi_n = \{x\colon f(x-\varphi(h)) \leqslant f(x-\varphi(h)-h) \text{ for } 0 < h < 1/n\}.$$

Then $A(f, 0)$ clearly equals the countable union of all the sets $A_n(f, 0) \equiv \Phi_n \cap A(f, 0)$.

Now, suppose that $A_n(f, 0)$ is of the second category. Then $A_n(f, 0)$ is of the second category in every open subinterval of some open interval I. But, according to the following lemma, this implies that $D_* f(x) \geqslant 0$ for each x in some subinterval J of I; that is, $A_n(f, 0) \cap J = \emptyset$, a contradiction. Hence, by proving the following lemma, we complete the proof of the theorem.

LEMMA. *Let $f\colon \mathbf{R} \to \mathbf{R}$ have the Baire property and let $\varphi \in P$. If Φ_n is of the second category on every open subinterval of the open interval I, then f is nondecreasing on some open subinterval J of I.*

Proof. Without loss of generality we assume $|I| < 1/n$ and choose a δ' such that $0 < \delta' < \delta$ and

$$|\varphi(\delta')| + \delta' < |I|/3. \tag{1}$$

Let J be the open interval with the same center as I, but having length δ'. It is this interval J on which f is nondecreasing. To see this, let a and b belong to J with $a < b$, let Σ be a residual subset of $\mathbf{R}$ such that $f|\Sigma$ is continuous, let $c \in \Sigma \cap (a, b)$, and let $\varepsilon > 0$. We will show that $f(a) \leqslant f(c) \leqslant f(b)$. Suppose L is an open interval in (a, b) with $c \in L$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{for all } x \in \Sigma \cap L.$$

Set $H = \{h\colon a+h \in L\}$ and $K = \{x\colon x = a+\varphi(h)+h,\ h \in H\}$. Clearly, K is an interval and $K \subseteq I$ because of (1). Now, $\{x\colon x = a+\varphi(h)+h,\ a+h \in L \cap \Sigma\}$ is residual in K, and Φ_n is of the second category in K. Hence, there are an $x_0 \in K \cap \Phi_n$ and an $h_0 \in H$ such that $x_0 = a+\varphi(h_0)+h_0$ and $a+h_0 \in L \cap \Sigma$. Then

$$0 \leqslant f(x_0-\varphi(h_0)) - f(x_0-\varphi(h_0)-h_0) = f(a+h_0) - f(a) \leqslant f(c)+\varepsilon - f(a),$$

and since this holds for each $\varepsilon > 0$, we have $f(a) \leqslant f(c)$. Next, let $H' = \{h\colon b-h \in L\}$ and $K' = \{x\colon x = b+\varphi(h),\ h \in H'\}$. As before, K' is an interval and $K' \subseteq I$. Also $\{x\colon x = b+\varphi(h),\ b-h \in L\}$ is residual in K' and Φ_n is of the second category in K'. Hence, there are an $x_1 \in K' \cap \Phi_n$ and an $h_1 \in H'$ such that $x_1 = b+\varphi(h_1)$ and $b-h_1 \in L \cap \Sigma$. Then

$$0 \leqslant f(x_1-\varphi(h_1)) - f(x_1-\varphi(h_1)-h_1) = f(b) - f(b-h_1) \leqslant f(b) - f(c) + \varepsilon.$$

Consequently, $f(c) \leqslant f(b)$ and the lemma is proved.

We are now in a position to state the main result of this section.

THEOREM 2. *Let f: $\boldsymbol{R} \to \boldsymbol{R}$ be measurable and let $\varphi \in S^*$. If f is φ-differentiable almost everywhere, then f is differentiable (almost everywhere and) nearly everywhere.*

Proof. As Sindalovskiĭ [6] points out, $f^\varphi(x)$ can be infinite only on a set of measure zero and, as a consequence, Theorem S guarantees that f is differentiable almost everywhere. Thus, f must have the Baire property, and Theorem 1 shows that f is differentiable nearly everywhere.

3. Monotonicity and related results. If we let f: $\boldsymbol{R} \to \boldsymbol{R}$ be the characteristic function of a point, and let $\varphi \in S^*$, we see that $f^\varphi \equiv 0$. So, the fact that f^φ is nonnegative need not imply that f is monotone unless something in addition to measurability is assumed for f. The Darboux property is a sufficient condition; indeed, membership in the class M_{-1}, introduced in [2], is necessary and sufficient as the next theorem shows.

Definition. A function f: $\boldsymbol{R} \to \boldsymbol{R}$ belongs to the *class* M_{-1} if f is measurable and if for each $x \in \boldsymbol{R}$ we have

$$\liminf_{t \to x} f(t) \leqslant f(x) \leqslant \limsup_{t \to x} f(t).$$

THEOREM 3. *Let $f \in M_{-1}$ and $\varphi \in S^*$. If f has a nonnegative φ-derivative at each point in $\boldsymbol{R}$, then f is nondecreasing on $\boldsymbol{R}$.*

Proof. Assume that $f^\varphi(x) > 0$ for each x. The general case then follows by considering the function $f(x) + \varepsilon x$ for each $\varepsilon > 0$.

Suppose there are two numbers a and b with $a < b$ and $f(a) > f(b)$. From Theorem S we see that f is continuous almost everywhere. Let x_0 be a point of continuity of f in (a, b) and let α be a number in $(f(b), f(a))$ different from $f(x_0)$. Then we are assured that one of the two sets $E^\alpha = \{x \in [a, b]: f(x) \leqslant \alpha\}$ or $E_\alpha = \{x \in [a, b]: f(x) \geqslant \alpha\}$ must contain a subinterval of (a, b). For definiteness, suppose E^α contains such a subinterval. (The other situation is handled analogously.) Let (c, d) be an interval in E^α such that

$$c = \inf\{x: (x, d) \subseteq E^\alpha\}. \tag{2}$$

Since $f^\varphi(c) > 0$, there is a number δ such that $0 < \delta < \delta_\varphi$ and

$$f(c - \varphi(h) - h) < f(c - \varphi(h)) \quad \text{for all } 0 < h < \delta. \tag{3}$$

Hence, $f(x) \leqslant \alpha$ for all $x \in (c - \varphi(\delta) - \delta, d) - \{c\}$. But since $f \in M_{-1}$, it follows that $f(c) \leqslant \alpha$; that is,

$$c \in E^\alpha. \tag{4}$$

Since $f(a) > \alpha$, we see that $c > a$ and that (2) conflicts with (3) and (4). This contradiction completes the proof.

Next we prove a slight generalization of this theorem which will be used later in this section.

COROLLARY 1. *Let $f \in M_{-1}$ and $\varphi \in S^*$. Suppose that f has a finite φ-derivative everywhere in $\boldsymbol{R}$ and $f^\varphi(x) \geqslant 0$ for almost every x in $\boldsymbol{R}$. Then f is nondecreasing.*

Proof. Let $E = \{x: f^\varphi(x) < 0\}$ and let g be a continuous nondecreasing function on $\boldsymbol{R}$ such that $g'(x) = +\infty$ for each $x \in E$. (Such a function g exists according to Theorem 6 in [5]). Let ε be a positive number and set $F(x) = f(x) + \varepsilon g(x)$. Theorem 3 indicates that F is nondecreasing, and since this holds for each positive ε, the result follows.

In the example given in the first paragraph of this section we saw an instance of a measurable function f for which $f^\varphi(x) \geqslant 0$ for every x and yet f was not monotone. However, there is clearly a continuous function g which agrees with f at "most" places, which is monotone, and which has the same φ-derivative as f. The next theorem shows that this situation is typical.

THEOREM 4. *Let $\varphi \in S^*$ and let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a measurable φ-differentiable function with $f^\varphi(x) \geqslant 0$ for each $x \in \boldsymbol{R}$. Then there are a set $\boldsymbol{T}$, whose complement is nowhere dense of measure zero, and a nondecreasing function $g\colon \boldsymbol{R} \to \boldsymbol{R}$ for which*

(i) *$f(x) = g(x)$ for each $x \in \boldsymbol{T}$,*

(ii) *$f^\varphi(x) = g^\varphi(x)$ for each $x \in \boldsymbol{R}$.*

Furthermore, if $f^\varphi(x)$ is finite for all x, then g is continuous on $\boldsymbol{R}$.

Proof. From Theorem 2 we know that f is differentiable almost everywhere and nearly everywhere. Since f is differentiable nearly everywhere, the theorem in [4] assures us that f is continuous except at a nowhere dense set of points. Let $\boldsymbol{T}$ be the set of points of continuity of f. Then the complement of $\boldsymbol{T}$ is nowhere dense of measure zero.

Let G denote the interior of $\boldsymbol{T}$. We first prove that $f|G$, the restriction of f to G, is nondecreasing, from which it readily follows that $f|\boldsymbol{T}$ is nondecreasing since the complement of G is nowhere dense and f is continuous at each point of $\boldsymbol{T}$. In order to show that $f|G$ is nondecreasing, there is no loss of generality in assuming that $f^\varphi(x) > 0$ for all x.

Let I be any component of G. Then $f|I$ is nondecreasing according to Theorem 3. Suppose there are two points ξ and η in G such that $\xi < \eta$ and $f(\xi) > f(\eta)$. Let (a, b) be the component of G containing ξ. If $\alpha = \sup\{f(x)\colon x \in (a, b)\}$, pick any γ such that $f(\eta) < \gamma < \alpha$ and note that, since $f^\varphi(b) > 0$, we have

$$\{x\colon f[b, x) \subseteq (\gamma, \infty)\} \neq \emptyset.$$

Now put

$$x_0 = \sup\{x\colon f[G \cap (b, \infty)] \subseteq (\gamma, \infty)\}. \tag{5}$$

Since $f^{\varphi}(x_0) > 0$, there is a δ such that $0 < \delta < \delta_{\varphi}$ and $f(x_0 - \varphi(h) - h) < f(x_0 - \varphi(h))$ for $0 < h < \delta$. However, $(x_0, x_0 - \varphi(\delta)) \cap G \neq \emptyset$, and hence there is an interval $I \subseteq (x_0, x_0 - \varphi(\delta))$ such that $f(I) \subseteq (-\infty, \gamma]$. Now, let $J = \{x_0 - \varphi(h) - h: x_0 - \varphi(h) \in I\}$ and note that $f(J) \subseteq (-\infty, \gamma]$. But this contradicts (5) since $J \cap G = \emptyset$. It follows that $f|G$ is nondecreasing.

Defining $g: \boldsymbol{R} \to \boldsymbol{R}$ by

$$g(x) = \inf\{f(t): t \in G \text{ and } t \geqslant x\}$$

we see that (i) is clearly satisfied. Now, let $x_0 \in \boldsymbol{R}$ and $\varepsilon > 0$. For each $h > 0$, there is an h^* with $0 < h^* < \varepsilon h$ such that $x_0 - \varphi(h + h^*)$ and $x_0 - \varphi(h + h^*) - h - h^*$ both belong to G. Then

$$\frac{g(x_0 - \varphi(h)) - g(x_0 - \varphi(h) - h)}{h} \leqslant \frac{g(x_0 - \varphi(h + h^*)) - g(x_0 - \varphi(h + h^*) - h - h^*)}{h}$$
$$= \frac{f(x_0 - \varphi(h + h^*)) - f(x_0 - \varphi(h + h^*) - h - h^*)}{h + h^*} \; \frac{h + h^*}{h}.$$

Hence $D^{\varphi} g(x_0) \leqslant f^{\varphi}(x_0)(1 + \varepsilon)$. Since this holds for each $\varepsilon > 0$, we have $D^{\varphi} g(x_0) \leqslant f^{\varphi}(x_0)$. Similarly, we can show $D_{\varphi} g(x_0) \geqslant f^{\varphi}(x_0)$, and so $g^{\varphi}(x_0)$ exists and equals $f^{\varphi}(x_0)$.

If $f^{\varphi}(x)$ is always finite, then clearly the nondecreasing function g can have no discontinuities.

In [3] Filipczak showed that if $f: \boldsymbol{R} \to \boldsymbol{R}$ is approximately continuous and has a symmetric derivative everywhere, then this symmetric derivative must belong to the Baire class one. Theorem 4 complements this result by yielding the following observation concerning measurable functions:

COROLLARY 2. *Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be measurable and $\varphi \in S^*$. If $f^{\varphi}(x)$ exists everywhere and is bounded either from above or from below, then there is a continuous function g such that $f(x) = g(x)$ except possibly on a nowhere dense set of measure zero and $f^{\varphi} \equiv g^{\varphi}$. In particular, f^{φ} belongs to the Baire class one.*

As another application of Theorem 4 we note that Corollaries 1 and 2 of this paper can be used to prove the following result. We omit the proof here since it is essentially identical to the proof given for the symmetric derivative in [2], Theorem 6.

THEOREM 5. *Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be measurable and have a finite φ-derivative for $\varphi \in S^*$. If f^{φ} is bounded from above or from below and is a Darboux function, then f^{φ} also has the Denjoy property; that is, for each interval (α, β), $\{x: f^{\varphi}(x) \in (\alpha, \beta)\}$ is either empty or has positive Lebesgue measure.*

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