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COMPLETION VARIETIES

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This paper contains an answer to a question posed by H. Höft in 1974 ([2]). To state the problem we need some definitions and notations (for those not defined here see e.g. [1]).

A signature is a pair $\langle F, n\rangle$ where F is a set and $n\colon F \to \omega$. We say that $f\in F$ is an $n(f)$-ary operation symbol. A partial algebra A is an algebra of signature $\langle F, n\rangle$ iff for every $f\in F$ it has an $n(f)$-ary partial operation f^A and no other fundamental operations. We assume that a signature $\langle F, n\rangle$ is fixed for all algebras considered below.

An inner extension of a partial algebra $A = (A, (f^A)_{f\in F})$ is a partial algebra $A' = (A, (f^{A'})_{f\in F})$ with $f^A \subset f^{A'}$ for every $f\in F$. A' is an inner completion of A if it is total. Let A^* be the set of all inner completions of a partial algebra A and for a class $\mathscr{K}$ of partial algebras let $\mathscr{K}^*$ be the class of all inner completions of algebras in $\mathscr{K}$. If $A = (A, (f^A)_{f\in F})$ and $B = (B, (f^B)_{f\in F})$ are two partial algebras, then a mapping $h\colon A \to B$ is a homomorphism of A into B iff for any $f\in F$ and $a_1, \ldots, a_{n(f)} \in A$ (clearly this sequence is empty when $n(f) = 0$), if $(a_1, \ldots, a_{n(f)}) \in \operatorname{Dom} f^A$, then $(h(a_1), \ldots, h(a_{n(f)})) \in \operatorname{Dom} f^B$ and $h(f^A(a_1, \ldots, a_{n(f)})) = f^B(h(a_1), \ldots, h(a_{n(f)}))$. If such h is onto, then B is said to be a homomorphic image of A. A subalgebra of a partial algebra $A = (A, (f^A)_{f\in F})$ is any partial algebra $B = (B, (f^B)_{f\in F})$ such that $B \subset A$ and for any $f\in F$ and $b_1, \ldots, b_{n(f)} \in B$, $f^B(b_1, \ldots, b_{n(f)})$ is defined in B (and then $f^B(b_1, \ldots, b_{n(f)}) = f^A(b_1, \ldots, b_{n(f)})$) iff $f^A(b_1, \ldots, b_{n(f)})$ is defined in A and belongs to B. Direct products of partial algebras are defined componentwise in a natural way.

For a class $\mathscr{K}$ of partial algebras let $H(\mathscr{K})$ be the class of all homomorphic images of partial algebras in $\mathscr{K}$, let $S(\mathscr{K})$ be the class of all isomorphic copies of subalgebras of partial algebras in $\mathscr{K}$ and let $P(\mathscr{K})$ be the class of all isomorphic copies of direct products of partial algebras in $\mathscr{K}$.

Höft's problem can now be stated as follows: characterize those partial algebras A for which the equality $HSP(A^*) = [HSP(A)]^*$ holds. It is easy to see that $HSP(A^*)$ is always a subclass of $[HSP(A)]^*$ and that the equality holds for all total algebras and for all one-element algebras; Höft furnishes an example of a non-trivial partial algebra for which it does not.

The first observation is that H and S are not relevant in the consideration of the problem.

LEMMA 1. *For any partial algebra* $A = (A, (f^A)_{f\in F})$, $S(A^*) = [S(A)]^*$.

Proof. Let $A_1 = (A_1, (f^{A_1})_{f\in F}) \in [S(A)]^*$. Define an inner completion $A_0 = (A, (f^{A_0})_{f\in F})$ of A so that for any $f \in F$, $f^A \subset f^{A_0}$ and $f^{A_0}|_{A_1} = f^{A_1}$; moreover, if $(a_1, \ldots, a_{n(f)}) \notin \mathrm{Dom} f^A \cup \mathrm{Dom} f^{A_1}$, then $f^{A_0}(a_1, \ldots, a_{n(f)}) = a_1$. A_1 is then a subalgebra of A_0, i.e., $A_1 \in S(A^*)$. On the other hand, if $A_1 = (A_1, (f^{A_1})_{f\in F})$ is a subalgebra of some $A_0 \in A^*$, then it is an inner completion of the partial algebra $(A_1, (f^A|_{A_1})_{f\in F}) \in S(A)$ and thus $A_1 \in [S(A)]^*$. □

LEMMA 2. *For any partial algebra* $A = (A, (f^A)_{f\in F})$, $H(A^*) = [H(A)]^*$.

Proof. $H(A^*) \subset [H(A)]^*$ is obvious, since a homomorphic image of an inner completion of A is a homomorphic image of A – and it is a total algebra. Let $B = (B, (f^B)_{f\in F}) \in H(A)$ and let $h: A \to B$ be an epimorphism. Furthermore, let $B_1 = (B, (f^{B_1})_{f\in F})$ be an inner completion of B. Define an algebra $A_1 = (A, (f^{A_1})_{f\in F})$ as follows: let $f \in F$ and $a_1, \ldots, a_{n(f)} \in A$; put

$$f^{A_1}(a_1, \ldots, a_{n(f)}) = \begin{cases} f^A(a_1, \ldots, a_{n(f)}) & \text{if } (a_1, \ldots, a_{n(f)}) \in \mathrm{Dom} f^A, \\ a & \text{otherwise} \end{cases}$$

where $a \in \overrightarrow{h^{-1}}(\{f^B(h(a_1), \ldots, h(a_{n(f)}))\})$. Then h is a homomorphism of a total algebra $A_1 \in A^*$ onto B_1, which proves $B_1 \in H(A^*)$. □

Unfortunately – and here is the essence of the problem – there is no "Lemma 3" for products. Consider $A = (\{a, b\}, f^A)$ with $n(f) = 1$ and such that $f^A(a) = b \notin \mathrm{Dom} f^A$. There are two inner completions of A, each of them satisfying the equation $f(x) = f^3(x)$ (which is also true in $P(A^*)$). Define $B = (\{a, b\}^2, f^B)$ so that $f^B(a, a) = (b, b)$, $f^B(b, b) = (a, b)$, $f^B(a, b) = (b, a)$ and $f^B(b, a) = (a, a)$. Then $B \in [P(A)]^*$ and $B \notin P(A^*)$, since $f^B(a, a) \neq (f^B)^3(a, a)$. Clearly $P(A^*) \subset [P(A)]^*$ is always true.

We can however characterize equations valid in $[P(A)]^*$. For a class $\mathscr{K}$ of total algebras (of a given signature) let $\mathrm{Eq}(\mathscr{K})$ be the set of all equations satisfied by all algebras in $\mathscr{K}$. If E is any set of equations, let $\bar{E}$ be its closure (the smallest set of equations containing E and closed under identities, symmetry, transitivity, substitution and the congruence condition). If A is a partial algebra, let $\mathrm{Eq}_{\mathrm{tot}}(A)$ be the set of all total equations in A, i.e., the set of all equations $p = q$ such that polynomials p^A and q^A induced by the terms p and q in A are total and equal. For any equation φ let φ_{L} and φ_{R} denote the left-hand and the right-hand side of φ, correspondingly. If φ is an equation in n variables, we write $(A, a_1, \ldots, a_n) \models \varphi$ to state that both φ_{L}^A and φ_{R}^A are defined on $a_1, \ldots, a_n$ and the elements $a_1, \ldots, a_n$ of A satisfy φ; $A \models \varphi$ means that $(A, a_1, \ldots, a_n) \models \varphi$ for all $a_1, \ldots, a_n$ in A. For a set of equations

E, $A \models E$ means $A \models \varphi$ for all $\varphi \in E$. We write $\vec{x_n}$ for a vector $(x_1, \ldots, x_n)$ of variables or elements of an algebra and $\vec{s_n}$ for a vector $(s_1, \ldots, s_n)$ of terms or polynomials in an algebra. Thus, e.g., $\vec{s_n}(\vec{x_k})$ denotes a vector $(s_1(x_1, \ldots, x_k), \ldots, s_n(x_1, \ldots, x_k))$ of terms (when the s_i are terms and the x_i are variables) or elements of an algebra (when the s_i are polynomials and the x_i are elements of the algebra)..

LEMMA 3. *For any partial algebra* $A = (A, (f^A)_{f \in F})$,

$$\mathrm{Eq}([P(A)]^*) = \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}.$$

Proof. If φ is a total equation in A, then it is also total in every algebra in $P(A)$ and thus in any inner completion of an algebra in $P(A)$. Therefore

$$\mathrm{Eq}_{\mathrm{tot}}(A) \subset \mathrm{Eq}([P(A)]^*)$$

and consequently

$$\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} \subset \overline{\mathrm{Eq}([P(A)]^*)} = \mathrm{Eq}([P(A)]^*)$$

(the latter being a closed set of equations).

For any term p define inductively $\mathrm{Fun}(p)$ as follows:

if $p = x$ for some variable x, then $\mathrm{Fun}(p) = 0$;

if $p = f(\vec{t_n})$ for some terms t_i $(i = 1, \ldots, n)$ and $f \in F$ then $\mathrm{Fun}(p) = \mathrm{Fun}(t_1) + \ldots + \mathrm{Fun}(t_n) + 1$.

Thus $\mathrm{Fun}(p)$ is the number of occurrences of operation symbols in p. For any equation φ set $\mathrm{Fun}(\varphi) = \mathrm{Fun}(\varphi_L) + \mathrm{Fun}(\varphi_R)$.

Suppose now $\mathrm{Eq}([P(A)]^*) - \overline{\mathrm{Eq}_{\mathrm{tot}}(A)} \neq \emptyset$ and let φ_0 be any equation in this set with a minimal number of occurrences of operation symbols, i.e., $\mathrm{Fun}(\varphi_0) \leqslant \mathrm{Fun}(\sigma)$ for any equation σ in $\mathrm{Eq}([P(A)]^*) - \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$, φ_0 being non-total in A, it contains some non-total subterms. Again let $f(\vec{t_{n(f)}}(\vec{x_n}))$ be a subterm of φ_0 (i.e., a subterm of φ_{0L} or φ_{0R}) non-total in A and such that $\mathrm{Fun}(f(\vec{t_{n(f)}}(\vec{x_n}))) \leqslant \mathrm{Fun}(p)$ for any non-total subterm p of φ_0. Thus the terms $t_i(\vec{x_n})$ are total in A. Let φ_0' be the equation obtained from φ_0 by substituting x_{n+1} for $f(\vec{t_{n(f)}}(\vec{x_n}))$ all throughout φ_0 (call φ_0' the simplified equation for φ_0). We may assume that for equations σ satisfying the same conditions as φ_0, $\mathrm{Fun}(\varphi_0') \leqslant \mathrm{Fun}(\sigma')$, where σ' is the simplified equation for σ. Then we have

LEMMA 3a. *If* $f(\vec{r_{n(f)}}(\vec{x_{n+1}}))$ *occurs in* φ_0', *then for some* i *with* $1 \leqslant i \leqslant n(f)$,

$$r_i(\vec{x_{n+1}}) = t_i(\vec{x_n}) \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}.$$

Proof. If the lemma were not true, we would be able to find an equation φ_1 "simpler" than φ_0. Indeed, suppose $f(\overrightarrow{r_{n(f)}}(\overrightarrow{x_{n+1}}))$ occurs in φ_0' and $r_i(\overrightarrow{x_{n+1}}) = t_i(\overrightarrow{x_n}) \in \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$ for all i, $1 \leqslant i \leqslant n(f)$. Let φ_1' be the equation obtained by substituting x_{n+1} for each occurrence of $f(\overrightarrow{r_{n(f)}}(\overrightarrow{x_{n+1}}))$ in φ_0' and let φ_1 be the equation obtained by substituting $f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))$ for each occurrence of x_{n+1} in φ_1' (then φ_1' is the simplified equation for φ_1). Clearly $\mathrm{Fun}(\varphi_1') < \mathrm{Fun}(\varphi_0')$. Moreover, since φ_1 is eventually built up from φ_0 by replacing every occurrence of $f(\overrightarrow{r_{n(f)}}(x_1, \ldots, x_n, f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))))$ in φ_0 by $f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))$ and

$$\mathrm{Fun}\big(f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))\big) \leqslant \mathrm{Fun}\Big(f\big(\overrightarrow{r_{n(f)}}(x_1, \ldots, x_n, f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n})))\big)\Big)$$

(either because the former is a subterm of the latter or by the assumption of the minimality of φ_0), then also $\mathrm{Fun}(\varphi_1) \leqslant \mathrm{Fun}(\varphi_0)$. Thus we shall have a contradiction if we prove that $\varphi_1 \in \mathrm{Eq}([P(A)]^*) - \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. Since $r_i(\overrightarrow{x_{n+1}}) = t_i(\overrightarrow{x_n}) \in \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$ for $i = 1, \ldots, n(f)$, we have

$$f(\overrightarrow{r_{n(f)}}(\overrightarrow{x_{n+1}})) = f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n})) \in \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$$

and consequently $\varphi_{1\mathrm{R}} = \varphi_{0\mathrm{R}}$ and $\varphi_{1\mathrm{L}} = \varphi_{0\mathrm{L}}$ are in $\overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. Thus $\varphi_0 \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$ implies $\varphi_1 \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. Furthermore, since $\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} \subset \mathrm{Eq}([P(A)]^*)$, we infer that for any $\boldsymbol{B} \in [P(A)]^*$, $\boldsymbol{B} \models \varphi_{1\mathrm{L}} = \varphi_{0\mathrm{L}}$ and $\boldsymbol{B} \models \varphi_{1\mathrm{R}} = \varphi_{0\mathrm{R}}$; hence $\boldsymbol{B} \models \varphi_1$ iff $\boldsymbol{B} \models \varphi_0$. This proves $\varphi_1 \in \mathrm{Eq}([P(A)]^*) - \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$, which contradicts the minimality of φ_0 in that set. □

We return to the proof of Lemma 3. Since $\varphi_0 \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$ and φ_0 is obtained from φ_0' by putting $f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))$ instead of x_{n+1}, then also $\varphi_0' \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. Moreover, $\mathrm{Fun}(\varphi_0') < \mathrm{Fun}(\varphi_0)$ and therefore $\varphi_0' \notin \mathrm{Eq}([P(A)]^*)$. Thus there exists an algebra $\boldsymbol{B} = (B, (f^{\boldsymbol{B}})_{f \in F}) \in [P(A)]^*$ and elements $b_1, \ldots, b_{n+1} \in B$ such that

$$(\boldsymbol{B}, b_1, \ldots, b_{n+1}) \not\models \varphi_0'. \tag{1}$$

Let $\{\overrightarrow{r^j_{n(f)}}(\overrightarrow{x_{n+1}}) \colon 1 \leqslant j \leqslant k\}$ for some $k \in \omega$ be the set of all $n(f)$-tuples of terms such that for every j with $1 \leqslant j \leqslant k$, $f(\overrightarrow{r^j_{n(f)}}(\overrightarrow{x_{n+1}}))$ occurs in φ_0' and $r_i^{j\boldsymbol{B}}(\overrightarrow{b_{n+1}}) = t_i^{\boldsymbol{B}}(\overrightarrow{b_n})$. It follows from Lemma 3a that for every j there exists an $i_j \in \{1, \ldots, n(f)\}$ such that $r^j_{i_j}(\overrightarrow{x_{n+1}}) = t_{i_j}(\overrightarrow{x_n}) \notin \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. Observe also that either $r^j_{i_j}(x_1, \ldots, x_n, f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n})))$ or $r^j_{i_j}(\overrightarrow{x_n})$ is a subterm of φ_0 as well as $f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))$, so $\mathrm{Fun}(r^j_{i_j}(\overrightarrow{x_{n+1}}) = t_{i_j}(\overrightarrow{x_n})) < \mathrm{Fun}(\varphi_0)$, proving $r^j_{i_j}(\overrightarrow{x_{n+1}}) = t_{i_j}(\overrightarrow{x_n}) \notin \mathrm{Eq}([P(A)]^*)$ – again by minimality of φ_0. Thus for every j with $1 \leqslant j \leqslant k$ there exists an algebra $\boldsymbol{C}_j = (C_j, (f^{\boldsymbol{C}_j})_{f \in F}) \in [P(A)]^*$ and elements $c^j_1, \ldots, c^j_{n+1} \in C_j$ such that

(2) $$(C_j, c_1^j, \ldots, c_{n+1}^j) \not\models r_{ij}^j(\overrightarrow{x_{n+1}}) = t_{ij}(\overrightarrow{x_n}).$$

Let A' be any inner completion of A and set $C = B \times A' \times C_1 \times \ldots \times C_k$; let $b_i' = b_i \times a_i \times c_i^1 \times \ldots \times c_i^k$ for $i = 1, \ldots, n$ and $b_{n+1}' = b_{n+1} \times a_1 \times c_{n+1}^1 \times \ldots \ldots \times c_{n+1}^k$, where $a_1, \ldots, a_n \in A$ are such that $f^A(\overrightarrow{t_{n(f)}^A(a_n)})$ is undefined in A (recall that $f(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n}))$ is not total in A). Then

$$C \in P([P(A)]^*) \subset [P(P(A))]^* = [P(A)]^*,$$

i.e., $C \in (A^\beta)^*$ for some ordinal β. (N.B., the proof can be done for the class of completions of finite products instead of arbitrary ones, hence β can be chosen as a finite ordinal. This is not essential for the sequel.) We infer by (1) that $(C, b_1', \ldots, b_{n+1}') \not\models \varphi_0'$. Define now an inner extension C_0 of the partial algebra A^β so that for any operation symbol $g \in F$,

$$g^{C_0}(d_1, \ldots, d_{n(g)}) = \begin{cases} g^{A^\beta}(d_1, \ldots, d_{n(g)}) & \text{if } (d_1, \ldots, d_{n(g)}) \in \operatorname{Dom} g^{A^\beta}, \\ g^C(d_1, \ldots, d_{n(g)}) & \text{if there are terms } s_1, \ldots, s_{n(g)} \text{ such that} \\ & g(\overrightarrow{s_{n(g)}}(\overrightarrow{x_{n+1}})) \text{ is a subterm of } \varphi_0' \text{ and} \\ & s_i^C(\overrightarrow{b_{n+1}'}) = d_i \text{ for } i = 1, \ldots, n(g), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Such definition of C_0 implies that both φ_{0L}' and φ_{0R}' are defined in C_0 for $b_1', \ldots, b_{n+1}'$ and $\varphi_{0L}'^{C_0}(\overrightarrow{b_{n+1}'}) = \varphi_{0L}'^{C}(\overrightarrow{b_{n+1}'})$ as well as $\varphi_{0R}'^{C_0}(\overrightarrow{b_{n+1}'}) = \varphi_{0R}'^{C}(\overrightarrow{b_{n+1}'})$. Thus

(3) $$(C_0, b_1', \ldots, b_{n+1}') \not\models \varphi_0'.$$

(Observe that C_0 is a minimal inner extension of A^β satisfying this condition.)

We shall prove that $f^{C_0}(\overrightarrow{t_{n(f)}^{C_0}}(\overrightarrow{b_n'}))$ is not defined in C_0. Indeed, by the choice of the a_i's, $(t_{n(f)}^A(\overrightarrow{a_n})) \notin \operatorname{Dom} f^A$ and so $(\overrightarrow{t_{n(f)}^{C_0}}(\overrightarrow{b_n'})) \notin \operatorname{Dom} f^{A^\beta}$. Moreover, if $s_1, \ldots, s_{n(f)}$ were terms such that $f(\overrightarrow{s_{n(f)}}(\overrightarrow{x_{n+1}}))$ occurs in φ_0' and $s_i^C(\overrightarrow{b_{n+1}'}) = t_i^C(\overrightarrow{b_n'})$ for all $i \in \{1, \ldots, n(f)\}$, then also $s_i^B(\overrightarrow{b_{n+1}'}) = t_i^B(\overrightarrow{b_n'})$ for all i. This means that for some j, $1 \leqslant j \leqslant k$,

$$(\overrightarrow{s_{n(f)}}(\overrightarrow{x_{n+1}})) = (\overrightarrow{r_{n(f)}^j}(\overrightarrow{x_{n+1}})).$$

By the definition of C_j we then have

$$s_{ij}^{C_j}(\overrightarrow{c_{n+1}^j}) = r_{ij}^{jC_j}(\overrightarrow{c_{n+1}^j}) \neq t_{ij}^{C_j}(\overrightarrow{c_{n+1}^j})$$

(by (2)) and consequently $s_{ij}^C(\overrightarrow{b_{n+1}'}) \neq t_{ij}^C(\overrightarrow{b_n'})$ contrarily to our assumption on the terms s_i. Thus $t_{n(f)}^{C_0}(\overrightarrow{b_n'}) \notin \operatorname{Dom} f^{C_0}$.

Now let C' be any inner completion of C_0 such that

$$f^{C'}\left(\overrightarrow{t_{n(f)}^{C'}}(\overrightarrow{b'_n})\right) = b_{n+1}. \tag{4}$$

Then $C' \in [P(A)]^*$, being an inner completion of A. It follows from (3) that $(C', b'_1, \ldots, b'_{n+1}) \not\models \varphi'_0$ and – recalling that φ_0 is obtained from φ'_0 by substituting $f\left(\overrightarrow{t_{n(f)}}(\overrightarrow{x_n})\right)$ for x_{n+1} and taking (4) into account $(C', b'_1, \ldots, b'_n) \not\models \varphi_0$. Hence $\varphi_0 \notin \mathrm{Eq}([P(A)]^*)$ – contrarily to the initial assumption. This completes the proof of the inclusion $\mathrm{Eq}([P(A)]^*) \subset \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$. $\square$

Now we are ready to state the main theorem:

THEOREM. *For any partial algebra A,*

$$\mathrm{HSP}(A^*) = [\mathrm{HSP}(A)]^* \quad \textit{iff} \quad \overline{\mathrm{Eq}_{\mathrm{tot}}(A)} = \mathrm{Eq}(A^*).$$

Proof. To prove the sufficiency it is enough to show that the equality $\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} = \mathrm{Eq}(A^*)$ implies $[HSP(A)]^* \subset HSP(A^*)$. Let $\boldsymbol{B}$ be any total algebra in the class $[HSP(A)]^*$. By Lemmas 1 and 2, $\boldsymbol{B} \in HS([P(A)]^*)$ and hence $\boldsymbol{B} \models \mathrm{Eq}([P(A)]^*)$. By Lemma 3, this is equivalent to $\boldsymbol{B} \models \overline{\mathrm{Eq}_{\mathrm{tot}}(A)}$, which by assumption implies $\boldsymbol{B} \models \mathrm{Eq}(A^*)$. Thus $\boldsymbol{B} \in HSP(A^*)$.

On the other hand, observe that $\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} \subset \mathrm{Eq}(A^*)$, since every total equation which holds in A also holds in any inner completion of A. Thus we have the following chain of inclusions and equalities, assuming $HSP(A^*) = [HSP(A)]^*$ and using Lemmas 1, 2 and 3:

$$\begin{aligned}\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} &\subset \mathrm{Eq}(A^*) = \mathrm{Eq}(HSP(A^*)) = \mathrm{Eq}([HSP(A)]^*) \\ &= \mathrm{Eq}(HS([P(A)]^*)) \subset \mathrm{Eq}([P(A)]^*) = \overline{\mathrm{Eq}_{\mathrm{tot}}(A)},\end{aligned}$$

thus proving $\overline{\mathrm{Eq}_{\mathrm{tot}}(A)} = \mathrm{Eq}(A^*)$. $\square$

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