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ON THE STRUCTURE OF THE STEINER TRIPLE SYSTEMS DERIVED FROM THE STEINER QUADRUPLE SYSTEMS

BY

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1. Introduction. A *Steiner triple system* (or, more simply, a *triple system*) is a pair (S, t), where S is a finite set and t is a collection of three-element subsets of S (called *triples*) such that each pair of distinct elements of S belong to exactly one triple of t. The number $|S|$ is called the *order of the triple system* (S, t) and it is well known that there is a Steiner triple system of order n if and only if $n \equiv 1$ or $3 \pmod 6$ [1]. A *Steiner quadruple system* (or *quadruple system*) is a pair $(Q, b(q))$, where Q is a finite set and $b(q)$ is a collection of four-element subsets of Q (called *quadruples* or *blocks*) such that any three distinct elements of Q belong to exactly one block of $b(q)$. As with triple systems the number $|Q|$ is called the *order of the quadruple system* $(Q, b(q))$. Hanani [2] proved in 1960 that the spectrum for quadruple systems consisted of the set of all positive integers $n \equiv 2$ or $4 \pmod 6$. For a quadruple system $(Q, b(q))$ and any element x in Q, denote $Q\setminus\{x\}$ by Q_x, and the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in b(q)$ by $b(q)(x)$. It is a routine matter to see that $(Q_x, b(q)(x))$ is a triple system called a *derived triple system* (DTS) of the quadruple system $(Q, b(q))$. Virtually nothing is known concerning DTS's of quadruple systems. In particular, essentially nothing is known concerning the following problem:

What are sufficient conditions for a given triple system to be a DTS of some quadruple system? (**P 958**)

Since there is (to within isomorphism) only one triple system of order 1, 3, 7 or 9, every triple system of one of these orders is a DTS. There are two triple systems of order 13 and Mendelsohn and Hung [12] have shown that both of these are DTS's. As far as the author can tell these two results are all that is known concerning whether or not a given triple system is a DTS. The purpose of this paper is to give some sufficient conditions for a triple system to be a DTS. Whether or not a given triple system is a DTS (besides being of interest in itself) is quite helpful in solving a large class of problems concerning triple systems. For example, if a triple system is a DTS, then it has a disjoint mate (a triple system on the same set of elements which is disjoint from it) [6]. The reader is referred to [3], [5]-[8] and [11] for a more detailed account of the connection between triple systems and quadruple systems.

2. Preliminaries. The main result in this paper is based on the following two recursive constructions. The first is a generalized singular direct product ([4] and [13]) for triple systems and the second is a generalized direct product for quadruple systems.

Let $(V, \boldsymbol{t})$ be any Steiner triple system and let $t_1, t_2, \ldots, t_r$ be the triples in $\boldsymbol{t}$. Let $(Q, \boldsymbol{q})$ be a triple system containing the subsystem $(P, \boldsymbol{p})$. Set $\bar{P} = Q \setminus P$ and, for each t_i belonging to $\boldsymbol{t}$, let $\otimes_i$ be a binary operation on $\bar{P}$ such that $(\bar{P}, \otimes_i)$ is a quasigroup. On the set $S = P \cup (\bar{P} \times V)$ define the following collection T of triples:

(1) $\{p, q, s\} \in T$ if and only if $\{p, q, s\} \in \boldsymbol{p}$;

(2) $\{p, (q, v), (s, v)\} \in T$ if and only if $p \in P$, $q \in \bar{P}$, $s \in \bar{P}$, $v \in V$ and $\{p, q, s\} \in \boldsymbol{q}$;

(3) $\{(p, w), (q, w), (s, w)\} \in T$ if and only if $p, q, s \in \bar{P}$ and $\{p, q, s\} \in \boldsymbol{q}$, $w \in V$;

(4) $\{(p, u), (q, v), (p \otimes_i q, w)\} \in T$ if and only if $p, q \in \bar{P}$, $t_i = \{u, v, w\}$, and $u < v < w$.

It is a routine matter to see that (S, T) is a Steiner triple system (see [9], for example). We will often denote the triple system just constructed by $(V, \boldsymbol{t}) \times Q(q, P, (\bar{P}, \otimes_i))$. It is important to note (because of what follows) that the quasigroups $(\bar{P}, \otimes_i)$ are not necessarily related and that, in fact, $(\bar{P}, \otimes_i)$ need have no special property other than, of course, being a quasigroup.

By a *3-skein* is meant a pair $(Q, \langle\, , ,\rangle)$, where Q is a finite set and $\langle\, , ,\rangle$ is a ternary operation on Q such that if in the equation $\langle x, y, z\rangle = w$ any three of x, y, z and w are given, then the remaining element is uniquely determined. Let $(Q, b(q))$ and $(V, b(v))$ be quadruple systems and $\{x, y, z, w\}_1$, $\{x, y, z, w\}_2, \ldots, \{x, y, z, w\}_k$ all blocks of $b(v)$. Let $(Q, \langle\, , ,\rangle_1), (Q, \langle\, , ,\rangle_2), \ldots, (Q, \langle\, , ,\rangle_k)$ be any k 3-skeins and define on the set $Q \times V$ the following collection $b(qv)$ of four-element subsets:

(1) For every block $\{a, b, c, d\} \in b(q)$ and every $w \in V$, $\{(a, w), (b, w), (c, w), (d, w)\} \in b(qv)$.

(2) For every two-element subset $\{a, b\}$ of Q and every two-element subset $\{u, w\}$ of V,

$$\{(a, u), (b, u), (a, w), (b, w)\} \in b(qv).$$

(3) For every block $\{a, b, c, d\} \in b(q)$ and every two-element subset $\{u, w\}$ of V, the following six subsets belong to $b(qv)$:

$$\{(a, u), (b, u), (c, w), (d, w)\}, \quad \{(a, w), (b, u), (c, u), (d, w)\},$$
$$\{(a, u), (b, w), (c, u), (d, w)\}, \quad \{(a, w), (b, u), (c, w), (d, u)\},$$
$$\{(a, u), (b, w), (c, w), (d, u)\}, \quad \{(a, w), (b, w), (c, u), (d, u)\}.$$

(4) For every block $\{x, y, z, w\}_i \in b(v)$ and every three (not necessarily distinct) elements p, q and s of Q,

$$\{(p, x), (q, y), (s, z), (\langle p, q, s\rangle_i, w)\} \in b(qv), \qquad \text{where } x < y < z < w.$$

In [5], it is shown that $(Q \times V, b(qv))$ is a quadruple system which we will sometimes denote by $(V, b(v)) \times Q(b(q), \langle\, , ,\rangle_i)$. As with the singular direct product for triple systems, the 3-skeins $(Q, \langle\, , ,\rangle_i)$ are not necessarily related. We will sometimes refer to the 3-skein $(Q, \langle\, , ,\rangle_i)$ as the 3-skein associated with $\{x, y, z, w\}_i$.

3. The structure of DTS's. Let $(P, \boldsymbol{p})$ and $(V, \boldsymbol{v})$ be DTS's, where $P = \{1, 2, \ldots, p\}$ and $V = \{1, 2, \ldots, v\}$. Let $(P^*, \boldsymbol{p}^*)$ and $(V^*, \boldsymbol{v}^*)$ be quadruple systems based on $P^* = \{0, 1, 2, \ldots, p\}$ and $V^* = \{0, 1, 2, \ldots, v\}$, respectively, such that

$$(P, \boldsymbol{p}) = (P_0^*, p^*(0)) \quad \text{and} \quad (V, \boldsymbol{v}) = (V_0^*, V^*(0)).$$

Further, let $(T, \boldsymbol{t})$ be the triple system of order 3 based on $T = \{1, 2, 3\}$, and let $(\overline{T}, \otimes)$ be the totally symmetric quasigroup defined by the following table:

$\otimes$	2	3
2	2	3
3	3	2

$(\overline{T}, \otimes)$

Set $(Q, \boldsymbol{q}) = (P, \boldsymbol{p}) \times T(t, \{1\}, (\overline{T}, \otimes))$. Then $\{2\} \times P$ equipped with the set of triples $\{(2, x), (2, y), (2, z)\}$ for all $\{x, y, z\} \in \boldsymbol{p}$ is a subsystem of $(Q, \boldsymbol{q})$ which is, of course, isomorphic to $(P, \boldsymbol{p})$. We denote this subsystem by $(2P, 2(\boldsymbol{p}))$.

THEOREM 1. *If $(P, \boldsymbol{p})$, $(V, \boldsymbol{v})$ and $(Q, \boldsymbol{q})$ are as above in this section, then any singular direct product $(V, v) \times Q(q, 2P, (\overline{P}, \otimes_i))$ is a DTS.*

Proof. We show that $(V, \boldsymbol{v}) \times Q(q, 2P, (\overline{P}, \otimes_i))$ is a DTS of a suitable generalized direct product of $(P^*, \boldsymbol{p}^*)$ and (V^*, v^*). We begin by noting that $\bar{P} = \{1\} \cup (\{3\} \times P)$. Let α be the mapping from P^* into $\bar{P}$ defined by $0\alpha = 1$ and $x\alpha = (3, x)$. For each triple $t_i = \{x, y, z\} \in \boldsymbol{v} = v^*(0)$, define a 3-skein $(P^*, \langle\, , \, , \rangle_i)$ by

$$\langle 0, a, b\rangle_i = \big((a\alpha)\otimes_i(b\alpha)\big)\alpha^{-1} \quad \text{for all } a, b \in P^*.$$

Associate any 3-skeins with the other blocks in v^* (the blocks not containing 0). Now let $(V^*, v^*) \times P^*(p^*, \langle\, , \, , \rangle_i)$ be the generalized direct product of (P^*, p^*) and (V^*, v^*) having the associated 3-skeins described above. Denote the quadruple system $(V^*, v^*) \times P^*(p^*, \langle\, , \, , \rangle_i)$ by (S, s), and the ordered pair $(0, 0)$ by 0^*. We claim that

$$\big(S_0^*, s(0^*)\big) = (V, \boldsymbol{v}) \times Q\big(q, 2P, (\bar{P}, \otimes_i)\big).$$

In (S, s) the blocks containing $(0, 0)$ are of the following form:

(1) $\{(0, 0), (x, 0), (y, 0), (z, 0)\}$, $\{0, x, y, z\} \in p^*$;

(2) $\{(0, 0), (x, 0), (0, y), (x, y)\}$, $x \neq y \in P$;

(3) $\{(0, 0), (x, 0), (y, w), (z, w)\}$, $\{(0, 0), (x, w), (y, 0), (z, w)\}$ and $\{(0, 0), (x, w), (y, w), (z, 0)\}$, where $\{0, x, y, z\} \in p^*$ and $w \neq 0$; and

(4) for every block $\{0, x, y, z\}_i \in v^*$ and any two (not necessarily distinct) elements a and b of P^*, $\{(0, 0), (a, x), (b, y), (\langle 0, a, b\rangle_i, z)\}$, where $0 < x < y < z$.

Therefore, in $\big(S_0^*, s(0^*)\big)$ the triples are of the following form:

(1′) $\{(x, 0), (y, 0), (z, 0)\}$, $\{x, y, z\} \in \boldsymbol{p} = p^*(0)$;

(2′) $\{(x, 0), (0, y), (x, y)\}$, $x \neq y \in P$;

(3′) $\{(x, 0), (y, w), (z, w)\}$, $\{(x, w), (y, 0), (z, w)\}$ and $\{(x, w), (y, w), (z, 0)\}$, where $\{x, y, z\} \in \boldsymbol{p} = p^*(0)$ and $w \neq 0$; and

(4′) for every triple $\{x, y, z\}_i \in \boldsymbol{v} = v^*(0)$ and any two (not necessarily distinct) elements a and b of $P = P_0^*$, $\{(a, x), (b, y), (\langle 0, a, b\rangle_i, z)\}$, where $x < y < z$.

In $(P, \boldsymbol{p}) \times T\big(t, \{1\}, (\bar{T}, \otimes)\big)$ the triples are of the following form:

(a) $\{1, (2, x), (3, x)\}$, all $x \in P$; and

(b) $\{(2, x), (2, y), (2, z)\}$, $\{(2, x), (3, y), (3, z)\}$, $\{(3, x), (2, y), (3, z)\}$, and $\{(3, x), (3, y), (2, z)\}$, all $\{x, y, z\} \in \boldsymbol{p}$.

Therefore, in $(V, v) \times Q\big(q, 2P, (\bar{P}, \otimes_i)\big)$ the triples are of the following form:

(c) $\{(2, x), (2, y), (2, z)\}$, $\{x, y, z\} \in \boldsymbol{p}$;

(d) $\big\{(1, v), (2, x), \big((3, x), v\big)\big\}$, all $x \in P$ and $v \in V$;

(e) $\big\{(2, x), \big((3, y), v\big), \big((3, z), v\big)\big\}$, $\big\{\big((3, x), v\big), (2, y), \big((3, z), v\big)\big\}$ and $\big\{\big((3, x), v\big), \big((3, y), v\big), (2, z)\big\}$, all $\{x, y, z\} \in \boldsymbol{p}$ and $v \in V$;

(f) for every triple $\{x, y, z\}_i \in \boldsymbol{v}$ and every two (not necessarily distinct) elements $a, b \in \bar{P}$, $\{(a, x), (b, y), (a \otimes_i b, z)\}$, where $x < y < z$.

Now let π be the mapping from $S_0^* = (P^* \times V^*) \setminus \{(0, 0)\}$ into $2\boldsymbol{P} \cup (\bar{P} \times V)$ defined by

$$(x, 0)\pi = (2, x) \quad \text{for all } x \in P,$$
$$(x, v)\pi = (xa, v) \quad \text{for all } v \in V,$$

where $0a = 1$ and $xa = (3, x)$ for $x \in P$.

It is a routine matter to see that π maps the triples of type (1′), (2′) and (3′) of $(S_0^*, s(0^*))$ onto the triples of type (c), (d) and (e) of $(V, \boldsymbol{v}) \times \times Q(q, 2P, (\bar{P}, \otimes_i))$. We now show that triples of type (4′) are preserved by π. So let

$$\{(a, x), (b, y), (\langle 0, a, b\rangle_i, z)\}$$

be any triple of type (4′) in $(S_0^*, s(0^*))$, where $\{x, y, z\}_i \in \boldsymbol{v} = v^*(0)$ and $x < y < z$. Then

$$\begin{aligned}\{(a, x)\pi, (b, y)\pi, (\langle 0, a, b\rangle_i, z)\pi\} \\ &= \{((aa)a^{-1}, x)\pi, ((ba)a^{-1}, y)\pi, (((aa)\otimes_i(ba))a^{-1}, z)\pi\} \\ &= \{(aa, x), (ba, y), ((aa)\otimes_i(ba), z)\}.\end{aligned}$$

Hence triples of type (4′) are mapped onto triples of type (f) proving that π is indeed an isomorphism. This completes the proof of the theorem.

We now restate Theorem 1 in a somewhat more appealing form:

THEOREM 2. *If $(P, \boldsymbol{p})$ and $(V, \boldsymbol{v})$ are DTS's, then so is any generalized singular direct product*

$$(V, \boldsymbol{v}) \times Q(q, P, (\bar{P}, \otimes_i)), \quad \textit{where } (Q, \boldsymbol{q}) = (P, \boldsymbol{p}) \times T(t, \{1\}, (\bar{T}, \otimes)).$$

4. Remarks. As mentioned in the introduction, every Steiner triple system of order 1, 3, 7, 9 or 13 is a DTS. These results along with Theorem 2 give an infinite class of DTS's for which the structure is known. A very interesting problem, unsolved as yet, is the construction of a quadruple system $(Q, b(q))$ having all of its DTS's mutually non-isomorphic. (**P 959**) In [7], the author gave the first general construction for quadruple systems having at least two non-isomorphic DTS's. Very recently, the author has shown that, for every positive integer t, there is a quadruple system having at least t mutually non-isomorphic DTS's [8]. Unfortunately, the construction used in [8] produces a quadruple system $(Q, b(q))$, where $|Q|$ is considerably larger than t. For example, if $t = 8$, the smallest order for which the construction in [8] gives a quadruple system $(Q, b(q))$ having at least 8 mutually non-isomorphic DTS's is $|Q| = 400$. Neither of the constructions used in [7] and [8] involve the structure of DTS's of quadruple systems (since nothing along these lines was previously known).

Possibly, the results in this paper can be used to construct the first quadruple system all of whose DTS's are non-isomorphic. For a list of some other problems involving quadruple systems and DTS's for which the results in this paper might prove useful, the reader is referred to [3], [6]-[8] and [10].

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