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ON STRONG RIESZ SETS

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Throughout this paper, $\boldsymbol{T}$ designates the circle group and $M(\boldsymbol{T})$ the convolution algebra of finite Borel measures on $\boldsymbol{T}$. The dual group of $\boldsymbol{T}$ is $\boldsymbol{Z}$, the additive group of integers. As usual, the Fourier-Stieltjes transformation is defined on $M(\boldsymbol{T})$ by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}d\mu(\theta) \quad (n \in \boldsymbol{Z}), \quad \text{where } \mu \in M(\boldsymbol{T}).$$

A subset S of $\boldsymbol{Z}$ is called a *Riesz set* if, for each $\mu \in M(\boldsymbol{T})$ with $\operatorname{supp}\hat{\mu} \subset S$, we infer that μ is absolutely continuous with respect to Lebesgue measure on $\boldsymbol{T}$. In [3], Meyer formulated the concept of a strong Riesz set as follows: $S \subset \boldsymbol{Z}$ is a strong Riesz set if the closure of S in $\boldsymbol{Z}$ is a Riesz set, where $\boldsymbol{Z}$ is armed with the relative topology of its Bohr compactification. The importance of the theory of strong Riesz sets resides in Theorem 2 of [3]: if S is a strong Riesz set and H is a Riesz set, then $S \cup H$ is a Riesz set. In propositions 4 and 3 of [3], Meyer proves that the set A of perfect squares, and the set P of prime integers are both strong Riesz sets. In the present paper, we extend both of these results. It is worth observing that, with proposition 5 of [3], one can construct strong Riesz sets of positive integers with density arbitrarily close to 1. On the other hand, there are sets of density zero which are not strong Riesz sets (see [1]). Therefore, it appears that it is the arithmetic properties and not the density properties which are pertinent in the study of strong Riesz sets.

THEOREM 1. *The set $L = \{a^2+b^2\colon\ a, b \in Z\}$ is a strong Riesz set.*

Proof. It will be sufficient to establish that, for each $n < 0$, there is a neighborhood of n (in $\boldsymbol{Z}$ with its relative Bohr topology) which misses L. This shows that the closure of L is bounded below and hence, by the classical F. and M. Riesz theorem, the closure of L is a Riesz set.

Now, $\boldsymbol{Z}$, with its relative Bohr topology, has the property that, for each $n \in \boldsymbol{Z}$, any arithmetic progression containing n is a neighborhood of n. So, for each $n < 0$, consider the neighborhood of n given by the arithmetic progression $O_n = \{n+4n^2k\colon\ k \in \boldsymbol{Z}\}$.

We show that $O_n \cap L = \emptyset$.

Since $n+4n^2k = (-n)(-1-4nk)$, we see that if $-1-4nk < 0$, then $n+4n^2k \notin L$. Also, $-1-4nk \neq 0, 1$ and so we may as well assume $-1-4nk > 1$. From [2], Theorem 7-8, we see that, for $l \geqslant 1$, $l \in L$ if and only if, for each prime divisor p of l such that $p \equiv 3 \pmod 4$, p occurs with even multiplicity in the canonical factorization of l. But

$$(-n, -1-4nk) = 1 \quad \text{and} \quad -1-4nk \equiv 3 \pmod 4,$$

and so $n+4n^2k \notin L$. Thus, $O_n \cap L = \emptyset$ and we are done.

COROLLARY 1. *If $\mu \in M(T)$ and* supp $\hat{\mu} \subset D$, *where D is the set of negative integers union the set of integers which are the sum of two squares, then μ is absolutely continuous with respect to Lebesgue measure.*

This corollary is of interest in view of 5.7 of [4], p. 226, since it is not known whether the set of integers which are the sums of two squares is a $\Lambda(1)$-set. In this connection see 4.6 of [4], p. 219.

THEOREM 2. *The set $S = \{k \in Z^+ :$ if $p|k$ and $n|k$ with $p \in P$ and $n > 1$, then $p \nmid n+1\}$ is a strong Riesz set.*

Proof. Let $n < -1$. We find an arithmetic progression containing n which misses S. Consider the arithmetic progression $\{n+3n^2k\}$ $(k \in Z)$. Since $n+3n^2k = -n(-1-3nk)$, we see that $n+3n^2k$ is not in S if $-1-3nk \leqslant 0$. Note also that $-1-3nk \neq 1$ for all k. Thus, it remains to show that $n+3n^2k \notin S$ if $-1-3nk > 1$. Since $n < -1$, it follows that $-n$ has a prime divisor, say, p. Then $p|(-1-3nk)+1$. Since $-1-3nk > 1$ and $-1-3nk$ is a factor of $n+3n^2k$ (as is p), it follows that $n+3n^2k \notin S$. This completes the proof.

COROLLARY 2. *The set of prime powers is a strong Riesz set.*

Proof. The set of prime powers is a subset of S.

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