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ON THE COEFFICIENTS OF UNIVALENT POLYNOMIALS

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INTRODUCTION

1. The present note concerns normalized polynomials, inverse to univalent functions in the unit disc, i.e. polynomials of the form

$$P_n(w) = w + C_2 w^2 + \ldots + C_n w^n \qquad (C_n \neq 0) \tag{1}$$

which are univalent in the largest domain D such that $0 \in D$ and $|P_n(w)| < 1$ for $w \in D$; this (simply connected) domain will be denoted by μ_{P_n}. Polynomials of the type (1) have been investigated in the general case by the first of the authors in [1].

Let $\mathfrak{P}_m$ denote the class of all described polynomials of the form (1) such that $n \leqslant m$ (m natural). The purpose of our note is to find the best possible estimates of $|C_2|$ in the class $\mathfrak{P}_2$, and of $|C_2|$ and $|C_3|$ in $\mathfrak{P}_3$. We find also all extremal functions in this problem. It seems that these results may give hints concerning the general case and, in this way, throw a certain light on the general coefficient problem.

2. In this note the following theorems are proved:

(i) *In the class $\mathfrak{P}_2$ the sharp estimate $|C_2| \leqslant \frac{1}{4}$ holds. All the extremal functions are given by the formula*

$$P_2^*(w) = w + \tfrac{1}{4} e^{i\vartheta} w^2,$$

where ϑ is real.

(ii) *In the class $\mathfrak{P}_3$ the sharp estimate $|C_2| \leqslant \frac{2}{9}\sqrt{3}$ holds. All the extremal functions are given by the formula*

$$P_3^*(w) = w + \tfrac{2}{9}\sqrt{3}\varepsilon e^{i\vartheta} w^2 + \tfrac{2}{27} e^{2i\vartheta} w^3,$$

where ϑ is real and $\varepsilon = 1$ or -1.

(iii) *In the class $\mathfrak{P}_3$ the sharp estimate $|C_3| \leqslant \frac{4}{27}$ holds. All the extremal functions are given by the formula*

$$P_3^*(w) = w + \tfrac{4}{27} e^{2i\vartheta} w^3,$$

where ϑ is real.

3. Our considerations are based on a general theorem obtained by the first of the authors (see [1], p. 20). This theorem may be formulated as follows:

If E is a holomorphic function of the complex variables $z_2, \ldots, z_k$ $(k \leqslant m)$ in a sufficiently large domain Δ, then in the class $\mathfrak{P}_m$ there exist extremal polynomials for which the functional $\operatorname{Re} E(C_2, \ldots, C_k)$ attains the maximum. Moreover, if

$$(2) \qquad P_n^*(w) = w + C_2^* w^2 + \ldots + C_n^* w^n \qquad (C_n^* \neq 0,\, k \leqslant n \leqslant m)$$

is one of such polynomials, $w_1, \ldots, w_l$ denote all zeros (different from each other) of the derived polynomial, and $\gamma_1, \ldots, \gamma_l$ — the multiplicity degrees of these zeros, respectively, then $w_1, \ldots, w_l$ lie on the boundary of the domain $\mu_{P_n^*}$ and there exist numbers $\varrho_\lambda > 0$ $(\lambda = 1, \ldots, l)$ satisfying the relations

$$(3) \qquad \operatorname{Res}_{w_\lambda}\{P_n^*(w)\Phi_k(w)/w^2 P_n^{*\prime}(w)\} = \varrho_\lambda \qquad (\lambda = 1, \ldots, l),$$

where

$$(4) \qquad \Phi_k(w) = \sum_{\varkappa=2}^{k} w^{1-\varkappa} \sum_{\nu=\varkappa}^{k} (\nu - \varkappa + 1) C_{\nu-\varkappa+1}^* E'_{z_\nu}(C_2^*, \ldots, C_k^*) \qquad (C_1^* = 1).$$

4. Obviously, from the quoted theorem we obtain only a necessary condition for the maximum of the functional $\operatorname{Re} E(C_2, \ldots, C_k)$. Then, in view of the aim of our note, it is worth while to remark that if a function f of one complex variable is holomorphic in a simply connected domain D, the derivative f' is different from zero in D, and the modulus $|f|$ is constant on the boundary of D, then f is univalent in D (see e.g. [2], p. 122). This implies that any polynomial satisfying the above formulated necessary condition for the extremum belongs to the class $\mathfrak{P}_m$. Hence, the problem is reduced to solving a certain system of equations and discussing these solutions.

5. Note finally that the modulus of the k-th coefficient $(k \leqslant n)$ of any polynomial of the form $z = e^{i\vartheta} P_n(e^{-i\vartheta} w)$, where P_n is fixed, and ϑ real and arbitrary, is the same, and that all these polynomials belong to the class $\mathfrak{P}_m$ provided $P_n \in \mathfrak{P}_m$.

§ 1. *ESTIMATION OF* $|C_2|$ *IN THE CLASS* $\mathfrak{P}_2$

6. According to Section 3 there exists an extremal polynomial of the form $P_2^*(w) = w + C_2^* w^2$ $(C_2^* \neq 0)$ for which the functional $\operatorname{Re} E(C_2) = \operatorname{Re} \log C_2$ attains the maximal value. Consequently, the estimate $|C_2| \leqslant |C_2^*|$ is the best possible in $\mathfrak{P}_2$. Moreover, in view of Section 5, we may, without any loss of generality, assume $P_2^*(w_1) = 1$. So, it remains to solve the system of algebraic equations

(5) $$w_1 + C_2^* w_1^2 = 1,$$

(6) $$1 + 2C_2^* w_1 = 0,$$

(7) $$\mathrm{Res}_{w_1}\{P_2^*(w)\Phi_2(w)/w^2 P_2^{*\prime}(w)\} = \varrho_1 > 0,$$

where $\Phi_2(w) = 1/C_2^* w$, as can be easily verified.

7. By solving the system of (5) and (6) we obtain $C_2^* = -\frac{1}{4}$ and $w_1 = 2$. Equation (7) yields $1/2C_2^{*2} w_1^3 = \varrho_1$, whence $\varrho_1 = 1$. In this way Theorem (i) is proved.

§ 2. *ESTIMATION OF* $|C_2|$ *IN THE CLASS* $\mathfrak{P}_3$

8. According to Section 3 there exists an extremal polynomial of the form $P_2^*(w) = w + C_2^* w^2$ $(C_2^* \neq 0)$ or $P_3^*(w) = w + C_2^* w^2 + C_3^* w^3$ $(C_3^* \neq 0)$ for which the functional $\mathrm{Re}\, E(C_2) = \mathrm{Re} \log C_2$ attains the maximal value. In the first case we obtain the same result as in Section 7. Thus, it remains to consider the other case. Since, in view of Section 5, we may, without loss of generality, assume $P_3^*(w_1) = \sigma$ and $P_3^*(w_2) = \bar{\sigma}$, where $|\sigma| = 1$, the problem is reduced to solving the system of algebraic equations

(8) $$w_1 + C_2^* w_1^2 + C_3^* w_1^3 = \sigma, \quad w_2 + C_2^* w_2^2 + C_3^* w_2^3 = \bar{\sigma} \quad (|\sigma| = 1),$$

(9) $$1 + 2C_2^* w_1 + 3C_3^* w_1^2 = 0, \quad 1 + 2C_2^* w_2 + 3C_3^* w_2^2 = 0,$$

(10) $$\mathrm{Res}_{w_1}\{P_3^*(w)\Phi_2(w)/w^2 P_3^{*\prime}(w)\} = \varrho_1 > 0,$$

$$\mathrm{Res}_{w_2}\{P_3^*(w)\Phi_2(w)/w^2 P_3^{*\prime}(w)\} = \varrho_2 > 0,$$

where $\Phi_2(w) = 1/C_2^* w$, as can be easily verified.

9. Obviously, C_2^*, w_1 and w_2 do not vanish. Hence, we may replace formulae (9) by

(11) $$C_3^* = 1/3w_1 w_2, \quad w_1 w_2 = -(w_1 + w_2)/2C_2^*.$$

Now, by adding and by multiplying the first and the second of the equations (8) together, we obtain, in view of (11) and $|\sigma| = 1$,

$$\tfrac{1}{3} C_2^* (w_1 + w_2)^2 + (w_1 + w_2) = \sigma + \bar{\sigma}$$

and

$$-\tfrac{1}{12}(w_1 + w_2)^2 - \tfrac{2}{9}(1/C_2^*)(w_1 + w_2) = 1,$$

respectively, whence

$$(12)\qquad \begin{gathered} w_1+w_2 = 9(4C_2^*+\sigma+\bar{\sigma}), \\ 432C_2^{*3}+216(\sigma+\bar{\sigma})C_2^{*2}+9\{3(\sigma+\bar{\sigma})^2+4\}C_2^*+8(\sigma+\bar{\sigma}) = 0. \end{gathered}$$

We consider successively two possibilities: $w_1 \neq w_2$ and $w_1 = w_2$.

10. Let $w_1 \neq w_2$. Then, after calculating the residues, the left-hand sides of (10) give $\sigma/3C_2^*C_3^*w_1^3(w_1-w_2)$ and $\bar{\sigma}/3C_2^*C_3^*w_2^3(w_2-w_1)$, respectively. Thus, by dividing both sides of the second of these equations by the first one, we have $\varrho_1 w_1^3/\varrho_2 w_2^3 = -\sigma/\bar{\sigma}$. Hence, in view of $|\sigma| = 1$, we obtain

$$(13)\qquad (w_1/w_2)^3+\mu\sigma^2 = 0 \qquad (\mu = \varrho_2/\varrho_1 > 0).$$

On the other hand, in view of (11), equations (8) yield $\frac{1}{6}w_1(3-w_1/w_2) = \sigma$ and $\frac{1}{6}w_2(3-w_2/w_1) = \bar{\sigma}$, respectively. Hence, in view of $|\sigma| = 1$,

$$(14)\qquad (w_1/w_2)^3-3(w_1/w_2)^2+3\sigma^2(w_1/w_2)-\sigma^2 = 0.$$

In order to eliminate w_1/w_2 we calculate the corresponding resultant of the system of (13) and (14):

$$(15)\qquad \Delta(\mu, \sigma) = -\sigma^6(27\mu\sigma^2+27\mu^2\sigma^{-2}+\mu^3-24\mu^2-24\mu+1) = 0.$$

Applying the condition $|\sigma| = 1$ again, we conclude that $\mu(\mu-1)\times \times(\sigma^2-\sigma^{-2}) = 0$. As $\mu > 0$, it remains to consider two particular cases $\mu = 1$ and $\sigma^4 = 1$ only.

11. Let first $w_1 \neq w_2$ and $\mu = 1$. Then (15) yields $\sigma^2+1/\sigma^2 = 46/27$. Hence $\sigma+1/\sigma = \frac{10}{9}\varepsilon_1\sqrt{3}$, where $\varepsilon_1 = 1$ or -1. Thus $\sigma = \frac{5}{9}\varepsilon_1\sqrt{3}+\frac{1}{9}\varepsilon_2\sqrt{6}i$, where $\varepsilon_2 = 1$ or -1. The second of equations (12) gives now

$$486C_2^{*3}+270\varepsilon_1\sqrt{3}C_2^{*2}+153C_2^*+10\varepsilon_1\sqrt{3} = 0.$$

Hence $C_2^* = -\frac{2}{9}\varepsilon_1\sqrt{3}$ or $C_2^* = -\frac{1}{6}\varepsilon_1\sqrt{3}+\frac{1}{18}\varepsilon_1\varepsilon_3\sqrt{3}i$, where $\varepsilon_3 = 1$ or -1. We choose the root $C_2^* = -\frac{2}{9}\varepsilon_1\sqrt{3}$ with the greatest modulus.

In turn, we derive w_1, w_2 and C_3^* from the first of the equations (12) and from (11). Since, as it can easily be verified, we have $\operatorname{Im} w_1 \neq \operatorname{Im} w_2$, it may be assumed, without loss of generality, that $\operatorname{Im} w_1 < \operatorname{Im} w_2$. Thus, we obtain $w_1 = \varepsilon_1\sqrt{3}-\frac{1}{2}\sqrt{6}i$, $w_2 = \varepsilon_1\sqrt{3}+\frac{1}{2}\sqrt{6}i$ and $C_3^* = \frac{2}{27}$. Consequently,

$$\sigma = P_3^*(w_1) = w_1+C_2^*w_1^2+C_3^*w_1^3 = \tfrac{5}{9}\varepsilon_1\sqrt{3}-\tfrac{1}{9}\sqrt{6}i,$$

whence $\varepsilon_2 = -1$.

The so obtained system of values $w_1 = -\varepsilon\sqrt{3}-\frac{1}{2}\sqrt{6}i$, $w_2 = -\varepsilon\sqrt{3}+ +\frac{1}{2}\sqrt{6}i$, $C_2^* = \frac{2}{9}\varepsilon\sqrt{3}$, $C_3^* = \frac{2}{27}$ ($\varepsilon = 1$ or -1) satisfies equations (8) and (9). Moreover, from (10) we obtain $\varrho_1 = \varrho_2 = \frac{1}{2} > 0$, and so, indeed, $\mu = 1$.

12. Let now $w_1 \neq w_2$ and $\sigma^4 = 1$. If $\sigma^2 = 1$, then (15) yields $\mu = -1 < 0$. Hence $\sigma^2 = -1$, and (15) gives $(\mu+1)(\mu^2-52\mu+1) = 0$. Hence $\mu = 26+15\varepsilon_1\sqrt{3}$, where $\varepsilon_1 = 1$ or -1. Since $\sigma = \varepsilon_2 i$, where $\varepsilon_2 = 1$ or -1, the second of the equations (12) yields $12C_2^{*3}+1 = 0$. It is easily seen that all roots of the last equation have the moduli less than the value $|\frac{2}{9}\varepsilon\sqrt{3}| = \frac{2}{9}\sqrt{3}$ obtained in Section 11.

13. Let finally $w_1 = w_2$. Then $\sigma = \bar{\sigma}$, whence, in view of $|\sigma| = 1$, we have $\sigma = \varepsilon$, where $\varepsilon = 1$ or -1. The second of the equations (12) yields $(3C_2^*+\varepsilon)^3 = 0$, and thus once more we obtain the value of C_2^* with the modulus less than that obtained in Section 11.

14. To complete the proof of (ii) we note that, according to Section 4, the extremal polynomial found in Section 11 belongs to the class $\mathfrak{P}_3$. As it is easily seen, this polynomial gives $|C_2^*|$ greater than that given by the extremal polynomial of the form $P_2^*(w) = w+C_2^*w^2$ $(C_2^* \neq 0)$ found in Section 7 (cf. Section 8).

§ 3. *ESTIMATION OF* $|C_3|$ *IN THE CLASS* $\mathfrak{P}_3$

15. According to Section 3 there exists an extremal polynomial of the form $P_3^*(w) = w+C_2^*w^2+C_3^*w^3$ $(C_3^* \neq 0)$ for which the functional $\operatorname{Re}E(C_2, C_3) = \operatorname{Re}\log C_3$ attains the maximal value. Consequently, the estimate $|C_3| \leqslant |C_3^*|$ is the best possible in $\mathfrak{P}_3$. Since, in view of Section 5, we may, without loss of generality, assume $P_3^*(w_1) = \sigma$ and $P_3^*(w_2) = \bar{\sigma}$, where $|\sigma| = 1$, the problem is reduced to solving the system of algebraic equations (8), (9) and

$$\begin{aligned} &\operatorname{Res}_{w_1}\{P_3^*(w)\Phi_3(w)/w^2P_3^{*\prime}(w)\} = \varrho_1 > 0, \\ &\operatorname{Res}_{w_2}\{P_3^*(w)\Phi_3(w)/w^2P_3^{*\prime}(w)\} = \varrho_2 > 0, \end{aligned} \tag{16}$$

where $\Phi_3(w) = 2C_2^*/C_3^*w+1/C_3^*w^2$, as can be easily verified.

16. Obviously, C_3^*, w_1 and w_2 do not vanish. Hence, we may replace formulae (9) by

$$C_2^* = -(w_1+w_2)/2w_1w_2, \qquad w_1w_2 = 1/3C_3^*. \tag{17}$$

Now, by adding and by multiplying the first and the second of the equations (8) together, we obtain, in view of (17) and $|\sigma| = 1$,

$$-\tfrac{1}{2}C_3^*(w_1+w_2)^3+(w_1+w_2) = \sigma+\bar{\sigma}$$

and

$$-\tfrac{1}{12}(w_1+w_2)^2+\tfrac{4}{27}(1/C_3^*) = 1,$$

respectively, whence

$$(54C_3^{*^2}+1)(w_1+w_2) = 9(\sigma+\sigma), \tag{18}$$
$$314928C_3^{*^3}-34992C_3^{*^2}+81\{9(\sigma+\bar{\sigma})^2-20\}C_3^*-16 = 0.$$

We consider succesively two possibilities: $w_1 \neq w_2$ and $w_1 = w_2$.

17. Let $w_1 \neq w_2$. Then, after calculating the residues, the left-hand sides of (16) give

$$\sigma(2C_2^*w_1+1)/3C_3^{*^2}w_1^4(w_1-w_2) \quad \text{and} \quad \bar{\sigma}(2C_2^*w_2+1)/3C_3^{*^2}w_2^4(w_2-w_1),$$

respectively. Then, by dividing both sides of the second of these equations by the first one and substituting C_2^* from (17), we have $\varrho_1 w_1^2/\varrho_2 w_2^2 = -\sigma/\bar{\sigma}$. Hence, in view of $|\sigma| = 1$, we obtain

$$(w_1/w_2)^2+\mu\sigma^2 = 0 \quad (\mu = \varrho_2/\varrho_1 > 0). \tag{19}$$

On the other hand, (8) and (11) imply (14). In order to eliminate w_1/w_2 we calculate the corresponding resultant of the system of (19) and (14):

$$\Delta(\mu, \sigma) = \sigma^5\{\mu(\mu-3)^2\sigma+(3\mu-1)^2\sigma^{-1}\} = 0. \tag{20}$$

Applying the condition $|\sigma| = 1$ again, we conclude that $(\mu-1)(\mu^2-14\mu+1)(\sigma-1/\sigma) = 0$. Thus, it is necessary to consider three particular cases: $\mu = 1$, $\mu^2-14\mu+1 = 0$ and $\sigma^2 = 1$.

18. Let first $w_1 \neq w_2$ and $\mu = 1$. Then (20) yields $\sigma+1/\sigma = 0$. Hence $\sigma = \varepsilon i$, where $\varepsilon = 1$ or -1. The second of the equations (18) gives now

$$78732C_3^{*^3}-8748C_3^{*^2}-405C_3^*-4 = 0.$$

Hence $C_3^* = 4/27$ or $C_3^* = -1/54$ (the double root). We choose the root $C_3^* = 4/27$ with the greater modulus.

In turn, we derive w_1, w_2 and C_2^* from the first of the equations (18) and from (17). Since, as it can easily be verified, we have $\operatorname{Im} w_1 \neq \operatorname{Im} w_2$, it may be assumed, without loss of generality, that $\operatorname{Im} w_1 < \operatorname{Im} w_2$. Thus, we obtain $w_1 = -\frac{3}{2}i$, $w_2 = \frac{3}{2}i$ and $C_2^* = 0$. Consequently, $\sigma = P_3^*(w_1) = w_1+C_2^*w_1^2+C_3^*w_1^3 = -i$, whence $\varepsilon = -1$.

The so-obtained system of values $w_1 = -\frac{3}{2}i$, $w_2 = \frac{3}{2}i$, $C_2^* = 0$ and $C_3^* = \frac{4}{27}$ satisfies equations (8) and (9). Moreover, from (16) we obtain $\varrho_1 = \varrho_2 = 1 > 0$, and so, indeed, $\mu = 1$.

19. Let next $w_1 \neq w_2$ and $\mu^2-14\mu+1 = 0$. Then $\mu = 7+4\varepsilon_1\sqrt{3}$, where $\varepsilon_1 = 1$ or -1, and from (20) we get $\sigma+1/\sigma = 0$. Hence $\sigma = \varepsilon_2 i$, where $\varepsilon_2 = 1$ or -1. Thus, analogously as in Section 18, the second of the equations (18) has the roots $C_3^* = 4/27$ and $C_3^* = -1/54$ (the double root). If $C_3^* = 4/27$, then, analogously as in Section 18, we have $\varrho_1 = \varrho_2 = 1$, whence $\mu = 1$ against $\mu = 7+4\varepsilon_1\sqrt{3}$. Thus, $C_3^* = -1/54$, and we obtain the value of C_3^* with the modulus less than that obtained in Section 18.

20. Let now $w_1 \neq w_2$ and $\sigma^2 = 1$. Then $\sigma = \varepsilon$, where $\varepsilon = 1$ or -1. Hence, from (20) we get $\mu = -1 < 0$ against $\mu = \varrho_2/\varrho_1 > 0$.

21. Let finally $w_1 = w_2$. Then $\sigma = \bar{\sigma}$, whence, in view of $|\sigma| = 1$, we have $\sigma = \varepsilon$, where $\varepsilon = 1$ or -1. The second of the equations (18) yields $(27C_3^*-1)^3 = 0$, and thus once more we obtain the value of C_3^* with the modulus less than that obtained in Section 18.

22. According to Section 4, the extremal polynomial found in Section 11 belongs to the class $\mathfrak{P}_3$. In this way the proof of (iii) is completed.

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