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NOWHERE DENSE DARBOUX GRAPHS

BY

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Introduction. Unless otherwise stated, the word "graph" shall refer to the graph of a real function, and if f is a point set in the plane, the X-projection of f is the set of all abscissas of points of f. The statement that the graph f is a Darboux graph means that if C is a connected subset of its X-projection, then $f(C)$ is connected. This class of graphs has been extensively investigated, and the expository article [2] by Bruckner and Ceder gives a detailed survey of work done in this field. There are many interesting examples of graphs having this and other rather unexpected and sometimes drastic properties. In many of these examples, the graphs are found to be dense in some circular region of the plane. It is the purpose of this paper to investigate those Darboux graphs which do not have this last property. In particular, the following will be shown:

THEOREM. *If f is a Darboux graph with X-projection an interval, then f is nowhere dense in the plane if and only if f is either continuous or else discontinuous only on a set of the first category.*

An example will be given to show that this theorem does not generalize to the more general Darboux transformations considered in [3]. Knaster and Kuratowski in [5] gave an example of a Darboux graph which is not a connected set, and Marcus in [6] proves with the aid of the Axiom of Choice that there actually exists a Darboux graph which is a totally disconnected set. An elementary example of a nowhere dense Darboux graph which is not connected will be given here, and it will be shown that Marcus's example can be altered to yield a nowhere dense, totally disconnected Darboux graph with X-projection an interval.

Proof of the Theorem. Suppose that f is a Darboux graph with X-projection an interval I. If there is a region R in which f is dense, then if J is an interval which is a subset of the X-projection of R, f would be discontinuous at each number in J. In this case, the set of numbers at which f is discontinuous would not be of the first category.

Now, suppose f is nowhere dense in the plane. Assume f is not continuous, and let M denote the set of all numbers x at which f is discontinuous. M must be of type F_σ, and a set of type F_σ is either of the first category or contains an interval. Suppose I_1 is an interval which is a subset of M. For each number t of I_1, let

$$p_t = \limsup_{x\to t} f(x) - \liminf_{x\to t} f(x),$$

if that is a number, and let $p_t = 1$, otherwise. Let I_2 be a subinterval of I_1 and p be a positive number such that the set $N = \{t \mid p_t > 2p\}$ is dense in I_2. If t is in I_2, and d is a positive number, there is an element u of N within $d/2$ of t, and there are numbers x and y within $d/2$ of u (thus within d of t) such that $|f(x)-f(y)| > p$. Therefore, if t is in I_2, $p_t \geqslant p$. Now, for each t in I_2, let z_t be a number such that

$$\limsup_{x\to t} f(x) - z_t \geqslant p/2 \quad \text{and} \quad z_t - \liminf_{x\to t} f(x) \geqslant p/2.$$

There must be a number K and a subinterval I_3 of I_2 such that $V = \{t \mid K < z_t < K+p/4\}$ is dense in I_3. Consider the open set $D = \{(x, y) \mid x$ is interior to I_3 and $K < y < K+p/4\}$. It will now be shown that f is dense in D. Let R be a circular region lying in D. Let t be an element of V in the X-projection X_R of R. There are numbers a and b in X_R such that

$$\left|\limsup_{x\to t} f(x) - f(b)\right| < p/4 \quad \text{and} \quad \left|\liminf_{x\to t} f(x) - f(a)\right| < p/4.$$

Thus, $f(b) > K+p/4$ and $f(a) < K$. Let w denote the ordinate of the center of R. There is a number c between a and b such that $f(c) = w$. Then (c, w) is a point of f in R. This means that f is dense in D, which is a contradiction.

Comments and Examples. It seems likely that the theorem above can be generalized to the real valued functions of several real variables. However, such is not the case for the general Darboux transformations from an Euclidean space X into a separable metric space X^* such as those considered by A. M. Bruckner and J. B. Bruckner in [3]. The following is a transformation T from the plane E_2 into E_2 such that (1) if C is a connected set in the plane, then $T(C)$ is connected, (2) T, considered as a subset of $E_2 \times E_2$, is nowhere dense, and (3) T is totally discontinuous.

Example 1. Let H be a collection of mutually exclusive, countable, dense subsets of the set E of all real numbers such that the union of all the sets in H is E, and let Q be a reversible transformation from H onto the set S of all lower semicontinuous graphs with X-projection E. Let f

be the graph such that if x is a number, and h is the set in H which contains x, and $g = Q(h)$, then $f(x) = g(x)$. The graph f is dense in the plane, and it follows from the theorem of [1] that if C is a connected number set, then the contraction of f to C is connected in the plane. Now let T be the transformation from E_2 into E_2 such that if (x, y) is a point of E_2, $T[(x, y)] = (x, f(x))$. If C is a connected subset of the plane, then the X-projection C_X of C is a connected number set, and $T(C)$ is the contraction of f to C_X and is therefore connected. Now, let U be a "region" in $E_2 \times E_2$, i.e. let $U = V \times W$ for a pair V and W of regions in the plane. Let V' and W' be subregions of V and W, respectively, such that the X-projections of V' and W' do not intersect. No point in V' has an image under T in W', so $V' \times W'$ is a "subregion" U' of U in $E_2 \times E_2$ that contains no ordered pair of T. Thus T, considered as a subset of $E_2 \times E_2$, is nowhere dense. T is obviously totally discontinuous. Notice that T is not even peripherally continuous in the sense of Hamilton [4], so that T is not a connectivity function.

Example 2. The following is an elementary example of a nowhere dense Darboux graph which is not connected. Let M be the "middle third" Cantor subset of $[0, 1]$. For each positive integer pair (n, j) such that $j \leqslant 2^{n-1}$, let $S_{n,j}$ be the j^{th} from the left component of $[0, 1] - M$ which has length $(1/3)^n$, and let $I_{n,j} = \mathrm{Cl}(S_{n,j})$. Let M' be the set of all elements x of M such that x not an end of any interval $I_{n,j}$. Let g be a non-decreasing graph with X-projection $[0, 1]$ which is continuous from the right, discontinuous only at the left ends of the intervals $I_{n,j}$, and constant over each $I_{n,j}$, and such that $g(0) = 0$ and $g(1) = 1$. Now, let f be a graph with the following properties:

(1) if n is an odd positive integer and $j \leqslant 2^{n-1}$ and $[a, b] = I_{n,j}$, then the contraction of f to $[a, b]$ is continuous and entirely above the contraction of g to $[a, b]$, reaching a maximum value of 1, and such that $f(b) - g(b) = f(a) - g(a) < 1/n$;

(2) if n is an even positive integer and $j \leqslant 2^{n-1}$ and $[a, b] = I_{n,j}$, then the contraction of f to $[a, b]$ is continuous and entirely below the contraction of g to $[a, b]$, reaching a minimum value of 0, and such that $g(b) - f(b) = g(a-) - f(a) < 1/n$,

(3) if x is in M', then $f(x) = 1$ if $x < 1/2$ and $f(x) = 0$ if $x > 1/2$.

The graph f is obviously nowhere dense, because it is continuous except for the elements of M. Let A be the set to which Z belongs if and only if Z is a point of the graph g or on a vertical interval with end points on $\mathrm{Cl}(g)$. A is an arc which separates f, so f is not connected. Now, suppose C is a connected subset of $[0, 1]$. If C is a subset of some interval $I_{n,j}$, then the contraction of f to C is continuous, so that $f(C)$ is connected. Suppose C contains an element of M'. Then C contains two elements x

and y of M', $x < y$. Let $d = g(y) - g(x)$, which is positive. Let n be an odd positive integer such that $1/n < d/4$ and j be a positive integer less than or equal to 2^{n-1} such that $I_{n,j}$ is a subset of $[x, y]$ and $g(t) < g(x) + d/4$ for each t in $I_{n,j}$. Let m be an even positive integer such that $1/m < d/4$ and k be a positive integer less than or equal to 2^{m-1} such that $I_{m,k}$ is a subset of $[x, y]$ and $g(t) > g(y) - d/4$ for each t in $I_{m,k}$. Then $f(I_{n,j}) = [u, 1]$ and $f(I_{m,k}) = [0, v]$, where $u < g(x) + d/2$ and $v > g(y) - d/2$. Therefore $f(C) = [0, 1]$, and f is a Darboux graph. Notice also that f is in the second Baire class.

Example 3. In Theorem 4 of [6] Marcus uses the axiom of choice to prove that there is a graph f with X-projection the set of all real numbers such that (1) f takes on each real value $\mathfrak{c}$ times ($\mathfrak{c}$ is the cardinality of the continuum) over each real perfect set, and (2) if a is a rational number different from 0, and b is a rational number, then the line L: $L(x) = ax + b$ and f do not intersect except possibly on the X-axis. Let f be such a graph. Now, for each interval J, let M_J denote the "middle third" Cantor subset of J, and let N_J denote the set of all numbers x such that x is in M_J but neither an end of J nor an end of any component of $J - M_J$. Let $I = [0, 1]$, and let $H_1 = N_I$. For each integer n greater than 1, let H_n denote the set of all numbers x which belong to N_J for some non-degenerate component J of $I - H_1 \cup H_2 \cup \ldots \cup H_{n-1}$. Now, let g be the graph with X-projection I defined as follows: if x is in I, and n is a positive integer such that x is in H_n, and $0 < f(x) \leqslant 1/n$, then $g(x) = f(x)$, otherwise $g(x) = 0$. Suppose C is a connected subset of I. There is a least positive integer n such that C and H_n intersect. There is a subset A of H_n and a countable set B such that $A \cup B$ is a perfect subset of C. Marcus's function f takes on each real value $\mathfrak{c}$ times over $A \cup B$, so $g(C) = [0, 1/n]$. Thus, g is a Darboux graph. Since g is continuous except on a set of the first category, g is nowhere dense in the plane. g has property (2) of Marcus's function and there is a dense subset of $[0, 1]$ over which g assumes only positive values, so that each two points of g are separated by the union of two rays L and L' having only their end point in common such that L is a vertical ray extending downward and L' is a subset of one of the lines $h(x) = ax + b$, where a is rational and different from 0 and b is rational.

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