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EQUIVALENCE THEOREMS FOR INFINITE PRODUCTS OF INDEPENDENT RANDOM ELEMENTS ON METRIC SEMIGROUPS

BY

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This paper is a continuation of [2]. The purpose is to establish the equivalence of convergence in law, in probability and with probability one for infinite products of independent random elements with values in a metric semigroup S (Theorems 3.1 and 3.2). Throughout this paper we assume, unless stated otherwise, that S satisfies conditions (R) and (L), i.e., for any compact $A, B \subseteq S$, $A^{-1}B$ and AB^{-1} are compact. Results of this paper generalize the earlier ones obtained by Loynes and others for random elements with values in topological groups.

1. Preliminaries. We use the terminology and notation introduced in [2]. We assume also that the reader is familiar with the standard semigroup concepts (see [4]). For information on topological semigroups (compact and locally compact semigroups, in particular) the reader is referred to [1], [11], and [12].

Now, let $\mu \in \tilde{S}$. A measure μ is called *idempotent* if $\mu\mu = \mu$. The following lemma describes the support of μ.

LEMMA 1.1. *Let S be a metric semigroup satisfying* (R). *If $\mu \in \tilde{S}$ is an idempotent measure, then its support is a completely simple subsemigroup.*

This lemma has been proved in [9] (Theorem 3.2) where S was assumed to be locally compact. The proof given there yields the following result: if S is a metric semigroup and $\mu \in \tilde{S}$ is an idempotent, then $C(\mu)$ (the support of μ) contains the minimal two-sided ideal K which is completely simple and dense in $C(\mu)$ (see [9], Remark 3). In order to prove that K is closed (and then $K = \bar{K} = C(\mu)$) it is sufficient to show that every closed subsemigroup G which is algebraically a group is, in fact, a topological group (the inversion in G is continuous) (see [9], Remark 3, and [1], p. 61). However, if G is a closed subsemigroup, then it satisfies also (R) and, therefore, the inversion in G is continuous whenever G is an algebraic group ([10], Proposition 1). By these remarks and by the proof of Theorem 3.2 in [9] we have also the following

LEMMA 1.2. *Let S be a metric semigroup satisfying* (R) *and let $\mu \in \tilde{S}$ be an idempotent. If the support $C(\mu)$ of μ is a left group, then $\mu s = \mu$ for every $s \in C(\mu)$.*

The next lemma is proved in [13] (Theorem 3.3) where S is assumed to be compact, however, the proof remains valid if S is assumed to be only metric.

LEMMA 1.3. *Let $\lambda \in \tilde{S}$. Suppose that the family*

$$\Pi = \{\lambda, \lambda^2, \ldots, \lambda^n, \ldots\}$$

is uniformly tight. Then the sequence

$$\varkappa_n = \frac{1}{n} \sum_{i=1}^{n} \lambda^i$$

converges in $\tilde{S}$ to a measure $m \in \tilde{S}$ such that

$$m\lambda = \lambda m = m = m^2.$$

Definition 1.1. Let S be an (algebraic) semigroup. $H \subseteq S$ is called *left* [*right*] *unitary* if $H^{-1}H \subseteq H$ [$HH^{-1} \subseteq H$]. If H is both left and right unitary, then it is called *unitary* (see [4], Vol. II, p. 55).

The following lemma is proved in [15], Lemma 2.1, and it is needed for our further considerations. Its proof is included here only for the sake of completeness.

LEMMA 1.4. *Let S be a semigroup having no non-trivial right-zero subsemigroups. Suppose that H is a left unitary subsemigroup of S with completely simple kernel K. Then H is a unitary subsemigroup of S.*

Proof. Observe first that the assumption that S has no non-trivial right-zero subsemigroups is equivalent to the following inclusion: $ee^{-1} \subseteq e^{-1}e$ for every $e \in E(S)$. From this assumption it follows also that the completely simple kernel K of H is a left group (see [4], § 2.7).

Now, let ab and b be some elements of H. If $k_1 \in K$, then $bk_1 \in K$. Let $k_2 \in bk_1K$ be such that $bk_1k_2 = e$, where $e^2 = e$ is the identity of the group bk_1K. Write $k = k_1k_2$. Then $bk = e$, so $abk = ae \in K$. Since K is left simple, there exists $x \in K$ such that $xae = e$ which means that $xa \in ee^{-1}$. Hence, by the observation given above, we obtain $xa \in e^{-1}e$. Thus ex, $(ex)a \in H$ and, since H is left unitary, it follows that $a \in H$, which completes the proof.

2. The equation $\mu\lambda = \mu$. Throughout this section, S will denote a metric semigroup satisfying condition (R).

We say that S *satisfies condition* (g) if for every $\mu \in \tilde{S}$ and for every $s \in S$ we have the following implication

$$(\mu s = \mu) \Rightarrow (xs = x \text{ for every } x \in C(\mu)).$$

The following theorem shows that (g) has a simple algebraic characterization:

THEOREM 2.1. *S satisfies* (g) *if and only if it has no non-trivial compact subgroups.*

Proof. Let us assume that S satisfies (g). Suppose, on the contrary, that there is a compact subgroup H such that $\operatorname{card} H > 1$. If m is the normed Haar measure on H, then, by the invariance of m, $ms = m$ for every $s \in H$, which contradicts (g), unless s is the identity of H.

Conversely, assume that S has no non-trivial compact subgroups. Let $\mu s = \mu$ for a measure $\mu \in \tilde{S}$ and a point $s \in S$. Let K be a compact subset of S such that $\mu(K) > 1/2$. By inequality (1.4) in [2] we have

$$s^n(K^{-1}K) \geqslant \mu s^n(K) + \mu(K) - 1 = 2\mu(K) - 1 > 0 \quad (n = 1, 2, \ldots).$$

It follows from (R) that $K^{-1}K$ is compact, so $\Gamma(s) = \overline{\{s, s^2, \ldots\}}$ is a compact (monothetic) semigroup. By [11], Lemma 3, its kernel K is a compact group. From the assumption it follows that K consists of one element, namely an idempotent e. Hence $\lim s^n = e$ (see [11], Lemma 3). Since $\mu s^n = \mu$ for $n = 1, 2, \ldots$, by the continuity of convolution we obtain $\mu e = \mu$. Now, $C(\mu)e$ is closed (e is an idempotent) and, therefore,

$$C(\mu)e = C(\mu). \tag{2.1}$$

If $x \in C(\mu)$, then, by (2.1), there exists $y \in C(\mu)$ such that $x = ye$. Hence $xe = yee = ye = x$. Since e is a zero in $\Gamma(s)$, we have

$$xs = (xe)s = x(es) = xe = x,$$

which completes the proof.

Now, we say that S *satisfies condition* (r) if for any $\mu, \lambda \in \tilde{S}$ we have

$$(\mu\lambda = \mu) \Rightarrow (\mu s = \mu \text{ for every } s \in C(\lambda)).$$

THEOREM 2.2. *S satisfies* (r) *if and only if it has no non-trivial right-zero subsemigroups.*

Proof. Let us assume that S satisfies (r). If $M = \{e, f\} \subseteq S$ is a right-zero subsemigroup ($e \neq f$), then $\mu\lambda = \mu$ for $\mu = \lambda = \frac{1}{2}(e+f)$ while $\mu e = e \neq \mu$. This proves that every right-zero subsemigroup in S is a trivial one.

Conversely, let us assume that $\mu\lambda = \mu$ for measures $\mu, \lambda \in \tilde{S}$. Then also $\mu\lambda^n = \mu$ for $n = 1, 2, \ldots$ and

$$\mu\varkappa_n = \mu, \quad \text{where } \varkappa_n = \frac{1}{n}\sum_{i=1}^{n} \lambda^i \ (n = 1, 2, \ldots). \tag{2.2}$$

From (1.4) in [2] we infer, exactly as in the proof of Theorem 2.1, that $\Pi = \{\lambda, \lambda^2, \ldots\}$ is uniformly tight. By Lemma 1.3 we see that $\lim \varkappa_n = m$, where $m \in \tilde{S}$ is such that $m\lambda = m = m^2$. By (2.2) and the continuity of convolution we obtain

$$\mu m = \mu. \tag{2.3}$$

In virtue of Lemma 1.1, $C(m)$, as the support of an idempotent measure, is a completely simple semigroup. From the assumption it follows that $C(m)$ is a left group. By Lemma 1.2 we have

$$ma = m \quad \text{for every } a \in C(m). \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$\mu a = \mu \quad \text{for every } a \in C(m). \tag{2.5}$$

Finally, let $a \in C(m)$ and $s \in C(\lambda)$. Then

$$as \in C(m)C(\lambda) \subseteq C(m),$$

and from (2.5) we obtain

$$\mu s = (\mu a)s = \mu(as) = \mu,$$

which completes the proof.

Remark 2.1. If S is a Hausdorff topological group, then Theorem 2.1 can easily be obtained from the result of Tortrat [14]. The fact that every topological group satisfies (r) has been observed by Choquet and Deny in [3], where S is assumed to be Abelian, and by Tortrat in [14] for general case. Hence, Theorem 2.2 generalizes the above results.

3. Equivalence theorems. Throughout this section, S will denote (unless stated otherwise) a metric semigroup satisfying (R) and (L) and having a left-subinvariant metric. We assume also that all considered random elements are defined on a complete probability space $(\Omega, \mathfrak{S}, P)$ and that their distributions are tight (they belong to $\tilde{S}$). Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random elements (i.r.e.) with values in S. The distribution of X_n is denoted by μ_n. In the remainder of this paper we use the notation

$$Y_m^n = X_m X_{m+1} \ldots X_n \quad (m \leqslant n), \qquad Y_n = Y_1^n,$$

and

$$\nu_m^n = \mu_m \mu_{m+1} \ldots \mu_n, \qquad \nu_n = \nu_1^n.$$

For further definitions and notation, see [2].

THEOREM 3.1. *Let S be a metric semigroup satisfying* (R) *and having a left subinvariant metric. The following statements are equivalent*:

(i) *If* $Y_n = X_1 X_2 \dots X_n$ *converges in distribution, then it converges in probability for every sequence* $(X_n)_{n \geqslant 1}$ *of i.r.e.*

(ii) *If* $(X_n)_{n \geqslant 1}$ *converges compositionally in distribution, then it converges compositionally in probability for every sequence* $(X_n)_{n \geqslant 1}$ *of i.r.e.*

(iii) S *satisfies* (g) *and* (r).

(iv) S *has no non-trivial compact subgroups or non-trivial right-zero subsemigroups.*

Proof. It is clear that (i) implies (ii). From Theorems 2.1 and 2.2 it follows that (iii) and (iv) are equivalent. Thus, it suffices to prove that (ii) implies (iv) and that (iii) implies (i).

Suppose that (ii) holds. If S contains a non-trivial compact subgroup H, then we can construct a sequence $(X_n)_{n \geqslant 1}$ of i.r.e. having an idempotent measure m, which is the normed Haar measure on H, as their distribution. Then $(X_n)_{n \geqslant 1}$ is obviously compositionally convergent in law, but $C(N_\infty) = H$ is not a left-zero semigroup, so, by Corollary 3.1 of [2], $(X_n)_{n \geqslant 1}$ does not converge compositionally in probability, which contradicts (ii). Next, if S contains a non-trivial right-zero subsemigroup M, and $e, f \in M$, $e \neq f$, then $\{e, f\}$ is also a right-zero semigroup. Arguing as above with $m = \frac{1}{2}(e+f)$ we also obtain a contradiction with (ii). Thus (ii) implies (iv).

Now, suppose that (iii) holds. Let $(X_n)_{n \geqslant 1}$ be a sequence of i.r.e. such that $Y_n = X_1 X_2 \dots X_n$ converges in distribution, i.e., $\lim \nu_n = \tilde{\nu}_1$ for a measure $\tilde{\nu}_1 \in \tilde{S}$. Then, by Lemma 2.2 of [2], $\tilde{\nu}_1 \lambda = \tilde{\nu}_1$ for every $\lambda \in \Lambda$. By the assumption, $xs = x$ for every $x \in C(\tilde{\nu}_1)$ and $s \in C(\lambda)$. Hence, $xC(\Lambda) = x$ for every $x \in C(\tilde{\nu}_1)$. From Theorem 3.1 of [2] it follows that Y_n converges in probability.

THEOREM 3.2. *The following statements are equivalent*:

(i) *If* $Y_n = X_1 X_2 \dots X_n$ *converges in probability, then it converges with probability one for every sequence* $(X_n)_{n \geqslant 1}$ *of i.r.e.*

(ii) *If* $(X_n)_{n \geqslant 1}$ *converges compositionally in probability, then it converges compositionally with probability one for every sequence* $(X_n)_{n \geqslant 1}$ *of i.r.e.*

(iii) S *satisfies* (r).

(iv) S *has no non-trivial right-zero subsemigroups.*

Proof. It is obvious that (i) implies (ii). From Theorem 2.2 it follows that (iii) and (iv) are equivalent.

Now, suppose that (ii) holds. If there exists a non-trivial right-zero subsemigroup $M = \{e, f\} \subseteq S$, then we can construct a sequence $(X_n)_{n \geqslant 1}$ of i.r.e., taking values in M, which converges to e in probability but not with probability one. Since $Y_k^n = X_n$ for every $k \leqslant n$, the sequence $(X_n)_{n \geqslant 1}$ converges compositionally in probability but not with probability one. Hence (ii) implies (iv).

Next, let us assume that (iv) holds. Let $(X_n)_{n \geqslant 1}$ be a sequence of i.r.e. such that $Y_n = X_1 X_2 \dots X_n$ converges in probability to an r.e. $\tilde{Y}_1$.

Let $\tilde{\nu}_1$ be the distribution of $\tilde{Y}_1$. By Theorem 3.1 of [2] we have

$$C(\Lambda) \subseteq x^{-1}x = H_x \quad \text{for every } x \in C(\tilde{\nu}_1) \tag{3.1}$$

($H_x = x^{-1}x = \{s; xs = x\}$). From (R) it follows that H_x is compact, thus it contains the completely simple kernel K. Since H_x is left unitary, from Lemma 1.4 it follows that it is a unitary subsemigroup of S, i.e., $H_x H_x^{-1} \subseteq H_x$. Hence, by (3.1) we obtain

$$C(\Lambda)C(\Lambda)^{-1} \subseteq H_x H_x^{-1} \subseteq H_x \quad \text{for every } x \in C(\tilde{\nu}_1).$$

By Corollary 4.1 of [2], Y_n converges with probability one.

COROLLARY 3.1. *Suppose that S does not contain any non-trivial compact subgroup or any non-trivial right-zero subsemigroup. If $(X_n)_{n\geqslant 1}$ is a sequence of i.r.e. such that $Y_n = X_1X_2\ldots X_n$ converges in distribution, then it converges also with probability one.*

Remark 3.1. Theorem 3.1 has been proved, with S being assumed a locally compact group, by several authors: Csiszár [5], Galmarino [6], Heyer [7] and others. Theorem 3.2 has been proved by Loynes [8] for S being a (Hausdorff) topological group. It is clear that our results hold without any additional assumption concerning $(\Omega, \mathfrak{S}, P)$ whenever S is separable (see the final remark in [2]).

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